8 Appendix: Proofs

Lemma 4. *Suppose Λ is a deterministic uniformly least favorable distribution for composite vs. simple test (H_0 vs. h_1) under $\mathcal{M} = (\mathcal{S}, \Theta, \vec{\pi})$. Then for any $n \in \mathbb{N}$, Λ is also a uniformly least favorable distribution for testing H_0 vs. h_1 under $\mathcal{M} = (\mathcal{S}^n, \Theta, \vec{\pi})$ with n i.i.d. samples.*

Proof: Let $\text{Spt}(\Lambda) = \{h_0^*\}$. For any $n \in \mathbb{N}$ and any $h_0 \in H_0$, we define a random variable $X_{n,h_0} : \mathcal{S}^n \to \mathbb{R}$, where for any $P_n \in \mathcal{S}^n$, $\Pr(P_n) = \pi_{h_0}(P_n) = \prod_{V \in P_n} \pi_{h_0}(V)$, and $X_{n,h_0}(P_n) = \log \text{Ratio}_{h_0^*,h_1}$. It follows that

$$X_{n,h_0} = \underbrace{X_{h_0} + X_{h_0} + \cdots + X_{h_0}}_{n}$$

By Lemma 3, for any $h_0 \in H_0$, $X_{h_0^*}$ weakly dominates X_{h_0}. Because first-order stochastic dominance is preserved under convolution [Deelstra and Plantin, 2014], we have that X_{n,h_0^*} weakly dominates X_{n,h_0}. The lemma follows after applying Lemma 3. $\square$

Remarks. Lemma 4 is an extension of Theorem 2.3 by Reinhardt Reinhardt [1961] to finite models. Reinhardt's theorem requires that for any constant t, with measure 0 we have $\pi_{h_0^*}(P) = t\pi_{h_1}(P)$. This is an important assumption in Reinhardt's proof because it assumes away cases with $\text{Ratio}(P) = k_\alpha$ so that the most powerful test is deterministic. Unfortunately, this assumption does not hold for finite models and we must deal with randomized tests.

Lemma 5 *Under a Mallows' model, for any φ, any $K \in \mathbb{N}$, any $a \in \mathcal{A}$, any $W \in \mathcal{L}(\mathcal{A})$, and any $C', C \subseteq \mathcal{A}$ such that C dominates C' w.r.t. W, we have $\pi_W(\{P : w_P(C' \succ a) \geq K\}) \leq \pi_W(\{P : w_P(C \succ a) \geq K\})$.*

Proof: We first prove the lemma for a special case where C and C' differ in only one alternative, that is, $|C - C'| = 1$. Let $c \in C$ such that $c \notin C'$. Let $c' \in C'$ such that $c' \notin C$. Because C dominates C' in W, we have $c \succ_W c'$.

Let $\mathcal{P} = \{P \in \mathcal{L}(\mathcal{A}) : w_P(C \succ a) \geq K\}$ and $\mathcal{P}' = \{P \in \mathcal{L}(\mathcal{A}) : w_P(C' \succ a) \geq K\}$. We define the following permutation $\mathcal{M}$ over $\mathcal{L}(\mathcal{A})$. For any $P \in \mathcal{L}(\mathcal{A})$, if $c \succ_P a \succ_P c'$ then $\mathcal{M}(P)$ is the ranking that is obtained from P by switching c and c'; otherwise $\mathcal{M}(P) = P$. Because $|C - C'| = 1$, it follows that for any $P \in \mathcal{P} - \mathcal{P}'$, we must have $c \succ_P a \succ_P c'$ and $(C - C') \succ_P a$. Therefore, $\mathcal{M}(\mathcal{P} - \mathcal{P}') = \mathcal{P}' - \mathcal{P}$.

We now prove that $\pi_W(\mathcal{P} - \mathcal{P}') > \pi_W(\mathcal{P}' - \mathcal{P})$. For any $P \in \mathcal{P} - \mathcal{P}'$, we have $c \succ_P a \succ_P c'$, which means that $\pi_W(P) \geq \pi_W(\mathcal{M}(P))/\varphi$ because $c \succ_W c'$. Therefore, $\pi_W(\mathcal{P} - \mathcal{P}') > \pi_W(\mathcal{P}' - \mathcal{P})$ because $\mathcal{M}(\mathcal{P} - \mathcal{P}') = \mathcal{P}' - \mathcal{P}$.

We have $\pi_W(\mathcal{P}) = \pi_W(\mathcal{P} \cap \mathcal{P}') + \pi_W(\mathcal{P} - \mathcal{P}') \geq \pi_W(\mathcal{P} \cap \mathcal{P}') + \pi_W(\mathcal{P}' - \mathcal{P}) = \pi_W(\mathcal{P}')$.

Therefore, the lemma holds for the case where $|C - C'| = 1$. For general C and C', because C dominates C', there exists a sequence of sets $C = C_0, C_1, \ldots, C_l = C'$ such that for all $0 \leq i \leq l-1$, (i) C_i dominates C_{i+1}; (ii) $|C_i - C_{i+1}| = 1$. It follows that $\pi_W(\{P : w_P(C \succ a) \geq K\}) \geq \pi_W(\{P : w_P(C_1 \succ a) \geq K\}) \geq \cdots \geq \pi_W(\{P : w_P(C' \succ a) \geq K\})$. $\square$

Theorem 2 (Characterization of all UMP non-winner tests under Mallows). *Given a Mallows' model $\mathcal{M}^{Ma}$ with $m \geq 2$ and $n \geq 2$, there exists a UMP test for $H_0 = L_{a \succ \text{others}}$ vs. H_1 for all $0 < \alpha < 1$ if and only if there exists $B \subseteq \mathcal{A}$ such that $H_1 \subseteq L_{B \succ a}$.*

Moreover, when $H_1 \subseteq L_{B \succ a}$, $f_{\alpha,a,B}$ defined in Theorem 1 is a UMP test.

Proof: The "if" part. We note that $f_{\alpha,a,B}$ does not depend on the orderings among alternatives in B in h_1. It follows that for all $h_1 \in H_1$, $f_{\alpha,a,B}$ is a level-α most powerful test for H_0 vs. $\{h_1\}$, which means that $f_{\alpha,a,B}$ is a UMP test.

The "only if" part. Suppose there exist B, B' such that $B \neq B'$ and there exist two rankings $h_1^1 = [B \succ a \succ \text{others}]$ and $h_1^2 = [B' \succ a \succ \text{others}]$ in H_1. W.l.o.g. suppose $B' - B \neq \emptyset$. Let α denote the number such that $K_\alpha = n|B| - 0.5$, $\Gamma_\alpha = 0$, and let $f_{\alpha,a,B}$ denote the most powerful test for H_0 vs. h_1^1 guaranteed by Theorem 1. Because K_α is not an integer, there does not exist P_n such that $w_{P_n}(B \succ a) = K_\alpha$. This means that $f_{\alpha,a,B}$ is the unique most powerful level-α test for H_0 vs. h_1^1. We observe that for any P_n, $f_{\alpha,a,B}(P_n)$ is either 0 or 1, and $f_{\alpha,a,B}(P_n) = 1$ if and only if a is ranked below B in all n rankings in P_n. It follows that $f_{\alpha,a,B}$ must be the unique level-α UMP test for H_0 vs. H_1.

By Theorem 1, any most powerful level-α test, in particular $f_{\alpha,a,B}$, must agree with $f_{\alpha,a,B'}$ except for the threshold cases $w_{P_n}(B' \succ a) = K'_\alpha$ for some K'_α. Choose arbitrary $b' \in B' - B$ and $b \in B$. Let P_n^* be composed of n copies of $[B \succ a \succ \text{others}]$ and let P'_n be composed of $n-1$ copies of $[b' \succ B \succ a \succ \text{others}]$ and one copy of $[b' \succ (B - \{b\}) \succ a \succ \text{others}]$. Because $w_{P_n^*}(B \succ a) = n|B| > K_\alpha$, we have $f_{\alpha,a,B}(P_n^*) = 1$. This means that the threshold K'_α for $f_{\alpha,a,B'}$ is no more than $w_{P_n^*}(B' \succ a) = n|B \cap B'|$. Because $n \geq 2$, we have $w_{P'_n}(B' \succ a) \geq n(|B \cap B'| + 1) - 1 > n|B \cap B'| = w_{P'_n}(B' \succ a)$, which means that $f_{\alpha,a,B}(P'_n) = 1$. However, $w_{P'_n}(B \succ a) = n|B| - 1 < n|B|$, which is a contradiction because for any profile P_n, $f_{\alpha,a,B}(P_n) = 1$ if and only if $B \succ a$ in all n rankings in P_n. □

Theorem 4. Let $\mathcal{M}^{\text{Ma}}$ denote a Mallows' model with $n = 1$, any $m \geq 4$, and any $\varphi < 1/m$. There exists $0 < \alpha < 1$ such that no level-α UMP test exists for $H_0 = (\mathcal{L}(\mathcal{A}) - H_1)$ vs. $H_1 = L_{a \succ \text{others}}$.

Proof: By Lemma 10, if a UMP test exists then $\bar{f}_{\alpha,a}$ is also a UMP test. Therefore, it suffices to prove that $\bar{f}_{\alpha,a}$ is not a level-α UMP test. To this end, we explicitly construct a test f and prove that the rankings assigned value 1 are more cost-effective than that under $\bar{f}_{\alpha,a}$.

Let $V_1, V_2, \ldots, V_m, V'_2 \in \mathcal{L}(\mathcal{A})$ denote $m+1$ rankings defined as follows. For any $j \leq m$, let $V_j = [a_j \succ \text{others}]$, where alternatives in "others" are ranked w.r.t. the increasing order of their subscripts. In other words, V_j is obtained from V_1 by raising alternative a_j to the top position. We let $V'_3 = [a_3 \succ a_1 \succ a_4 \succ a_2 \succ \text{others}]$.

We consider the following critical function f. For any $V \in \mathcal{L}_{a \succ \text{others}}$, we let $f(V) = 1$. For any V_j with $j \neq 3$, let $f(V_j) = 1$. We then let $f(V_3) = f(V'_3) = \frac{1+\varphi^m}{1+\varphi}$. Let α denote the size of f at V_2. That is, $\alpha = \text{Size}(f, V_2)$. Let $T = \pi_{V_2}(\mathcal{L}_{a \succ \text{others}})$. It follows that

$$\begin{aligned}
&\alpha - T \\
\propto&\varphi^0 + \frac{1+\varphi^m}{1+\varphi}(\varphi^{\text{KT}(V_2,V_3)} + \varphi^{\text{KT}(V_2,V'_3)}) + \sum_{j=5}^{m} \varphi^{\text{KT}(V_2,V_j)} \\
=&1 + \frac{1+\varphi^m}{1+\varphi}(\varphi^3 + \varphi^4) + \varphi^4 + \sum_{j=5}^{m} \varphi^{\text{KT}(V_2,V_j)} \\
>&1 + \varphi^3 + \varphi^4 + \varphi^5
\end{aligned}$$

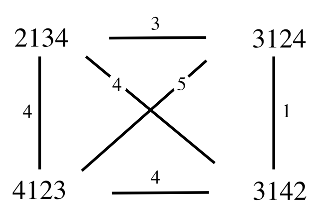

Figure 1: Kentall-Tau distance for some rankings over four alternatives.

For any $j, j^* \geq 2$ such that $j \neq j^*$, it is not hard to verify that $\text{KT}(V_j, V_{j^*}) = j + j^* - 2$. Moreover, $\text{KT}(V_3, V'_3) = 1$, $\text{KT}(V_2, V'_3) = 4$, $\text{KT}(V_4, V'_3) = 4$, and for any $j \geq 5$, we have $\text{KT}(V'_3, V_j) = j + 2$. Therefore, we have the following calculations of $\text{Size}(f, V_3)$, $\text{Size}(f, V'_3)$, and $\text{Size}(f, V_4)$ (see Figure 1 for distances between V_2, V_3, V'_3, V_4). We note that $T = \pi_{V_2}(\mathcal{L}_{a \succ \text{others}}) = \pi_{V_3}(\mathcal{L}_{a \succ \text{others}}) = \pi_{V'_3}(\mathcal{L}_{a \succ \text{others}}) = \pi_{V_4}(\mathcal{L}_{a \succ \text{others}})$ due to symmetry.

$$\text{Size}(f, V_3) - T \propto \varphi^3 + \tfrac{1+\varphi^m}{1+\varphi}(1+\varphi) + \varphi^5 + \textstyle\sum_{j=5} \varphi^{\text{KT}(V_3,V_j)} \leq 1 + \varphi^3 + (m-3)\varphi^5$$

$$\text{Size}(f, V'_3) - T \propto \varphi^4 + \tfrac{1+\varphi^m}{1+\varphi}(1+\varphi) + \varphi^4 + \textstyle\sum_{j=5} \varphi^{\text{KT}(V'_3,V_j)} \leq 1 + 2\varphi^4 + (m-4)\varphi^6$$

$$\text{Size}(f, V_4) - T \propto \varphi^4 + \tfrac{1+\varphi^m}{1+\varphi}(\varphi^4 + \varphi^5) + 1 + \textstyle\sum_{j=5} \varphi^{\text{KT}(V_4,V_j)} \leq 1 + 2\varphi^4 + (m-4)\varphi^7$$

For any other $h'_0 \in H_0$, we have $\text{Size}(f, h'_0) - T \leq m\varphi$. Because $\varphi < 1/m$, we have $\text{Size}(f) = \alpha$. Let P denote a profile that is composed of $\{V_2, V_4, \ldots, V_m\} \cup \frac{1+\varphi^m}{1+\varphi}\{V_3, V'_3\}$. We next prove that $\text{Ratio}_{V_2,V_1}(P) > \text{Ratio}_{V_2,V_1}(T_{m-2})$.

Let $Z_m = \prod_{l=1}^{m} \frac{1-\varphi^m}{1-\varphi}$ denote the Mallows normalization factor for m alternatives. We have

$$\begin{aligned}
&\text{Ratio}_{V_2,V_1}(T_{m-2}) = \frac{\pi_{V_1}(T_{m-2})}{\pi_{V_2}(T_{m-2})} \\
=&\frac{\varphi Z_{m-1}}{Z_{m-2} + \varphi^2(Z_{m-1} - Z_{m-2})} \\
=&\frac{\varphi \frac{Z_{m-1}}{Z_{m-2}}}{1 + \varphi^2(\frac{Z_{m-1}}{Z_{m-2}} - 1)} = \frac{\varphi + \varphi^2 + \cdots + \varphi^{m-1}}{1 + \varphi^3 + \varphi^4 + \cdots + \varphi^m} < \frac{1}{\varphi}
\end{aligned}$$

$$\begin{aligned}
&\text{Ratio}_{V_2,V_1}(P) = \frac{\varphi + \varphi^2 + \cdots + \varphi^{m-1} + \varphi^{m+2}}{1 + \varphi^3 + \varphi^4 + \cdots + \varphi^m + \varphi^{m+3}} \\
>&\frac{\varphi + \varphi^2 + \cdots + \varphi^{m-1}}{1 + \varphi^3 + \varphi^4 + \cdots + \varphi^m} \\
=&\text{Ratio}_{V_2,V_1}(T_{m-2})
\end{aligned}$$

We note that $\text{Size}(\bar{f}_{\alpha,a}, V_2) = \alpha$. This means that $\text{Power}(\bar{f}_{\alpha,a}, V_1) = \pi_{V_1}(T_{m-1}) + \alpha\text{Ratio}_{T_2,T_1}(T_{m-2}) < \pi_{V_1}(T_{m-1}) + \alpha\text{Ratio}_{T_2,T_1}(P) = \text{Power}(f, V_1)$. This means that $\bar{f}_{\alpha,a}$ is a not a level-α UMP. The theorem follows after Lemma 10. □

Theorem 5. Let $\mathcal{M}^{\text{Ma}}$ denote a Mallows' model with $n = 1$ and any $m \geq 4$. There exists $\epsilon > 0$ such that for any $\varphi > 1 - \epsilon$ and any α, $\bar{f}_{\alpha,a}$ is a UMP test for $H_0 = (\mathcal{L}(\mathcal{A}) - H_1)$ vs. $H_1 = L_{a\succ\text{others}}$.

Proof: We first verify that when $K_\alpha = m - 1$, $\bar{f}_{\alpha,a}$ is a UMP test. For any $h_1 \in H_1$, let $h_0^* \in H_0$ denote the ranking that is obtained from h_1 by moving a down for one position. It is not hard to check that for any $V \in \mathcal{L}(\mathcal{A})$, $\text{Ratio}_{h_0^*,h_1}(V) \leq 1/\varphi$, and for all $V \in H_1$ we have $\text{Ratio}_{h_0^*,h_1}(V) = 1/\varphi$. This means that for any level-α test for H_0 vs. h_1, the power cannot be more than α/φ. We note that $\bar{f}_{\alpha,a}$ is a level-α test whose power is exactly α/φ. This means that for all $h_1 \in H_1$, $\bar{f}_{\alpha,a}$ is a most powerful test for H_0 vs. h_1. Therefore, when $K_\alpha = m - 1$, $\bar{f}_{\alpha,a}$ is a UMP test.

For any α such that $K_\alpha \leq m - 2$, we will prove that for any $h_1 \in H_1$, $\bar{f}_{\alpha,a}$ is a most powerful level-α test for H_0 vs. h_1. This is done in the following steps. Step 1. Find a least favorable distribution $\Lambda_\alpha^{h_1}$ whose support is the set of all rankings where a is ranked at the second position. Step 2. Verify that $\bar{f}_{\alpha,a}$ is the likelihood ratio test w.r.t. $\Lambda_\alpha^{h_1}$, and step 3. verify that the two conditions in Lemma 2 holds for $\Lambda_\alpha^{h_1}$.

Step 1. The main challenge is that in general there does not exist a uniformly least favorable distribution. For different α we define different $\Lambda_\alpha^{h_1}$ as follows. For any α, we let s_α denote the smallest Borda score of the ranking V such that $\bar{f}_{\alpha,a}(V) > 0$. We have that $s_\alpha \leq m - 2$. Let the support of $\Lambda_\alpha^{h_1}$ be T_{m-2}, which is the set of rankings where a is ranked at the second position. We will solve the following system of linear equations to determine $\Lambda_\alpha^{h_1}$. For any $h_0^* \in T_{m-2}$ there is a variable $x[h_0, s_\alpha]$.

$$\forall V \in T_{s_\alpha}, \sum_{h_0^* \in T_{m-2}} \text{Ratio}_{h_0^*,h_1}^{-1}(V) \cdot x[h_0^*, s_\alpha] = m \qquad (\text{LP}_{s_\alpha}^{h_1})$$

We note that as $\varphi \to 1$, $\text{Ratio}_{h_0^*,h_1}^{-1}(V) = \frac{\pi_{h_0^*}(V)}{\pi_{h_1}(V)} = \varphi^{\text{KT}(h_0^*,V) - \text{KT}(h_1,V)} \to 1$. Because there are m variables and m equations, as $\varphi \to 1$ the solution to $\text{LP}_{s_\alpha}^{h_1}$ converges to $\vec{1}$. Therefore, there exists $\epsilon > 0$ such that for all $\varphi > 1 - \epsilon$, the linear systems $\{\text{LP}_s^{h_1} : s \leq m - 1, h_1 \in H_1\}$ all have strictly positive solutions. Let $\{x^*[h_0^*, s_\alpha] | V \in T_{s_\alpha}\}$ denote a solution to $\text{LP}_{s_\alpha}^{h_1}$. For any $h_0^* \in T_{m-2}$, we let $\Lambda_\alpha^{h_1}(h_0^*) = \frac{x^*[h_0^*, s_\alpha]}{\sum_{h_0 \in T_{m-2}} x^*[h_0, s_\alpha]}$.

Step 2. To simplify notation we let $\text{LR}_\alpha = \text{LR}_{\alpha,\Lambda_\alpha^{h_1},h_1}$ denote the likelihood ratio test and let $\text{Ratio} = \text{Ratio}_{\Lambda_\alpha^{h_1},h_1}$ denote the likelihood ratio function w.r.t. distribution $\Lambda_\alpha^{h_1}$ for H_0 vs. h_1. To prove $\text{LR}_\alpha = \bar{f}_{\alpha,a}$, we first prove that for any $V \in \mathcal{L}(\mathcal{A})$ where a is not ranked at the bottom position, $\text{Ratio}(V) > \text{Ratio}(\text{Down}_a^1(V))$, where we recall that

$\text{Down}_a^1(V)$ is the ranking obtained from V by moving a down for one position.

$$\begin{aligned}
&\frac{\sum_{h_0^*\in T_{m-2}} \Lambda_\alpha^{h_1}(h_0^*)\cdot \pi_{h_0^*}(\text{Down}_a^1(V))}{\sum_{h_0^*\in T_{m-2}} \Lambda_\alpha^{h_1}(h_0^*)\cdot \pi_{h_0^*}(V)}\\
=&\frac{\sum_{h_0^*\in T_{m-2}} \Lambda_\alpha^{h_1}(h_0^*)\cdot \varphi^{\text{KT}(h_0^*,\text{Down}_a^1(V))}}{\sum_{h_0^*\in T_{m-2}} \Lambda_\alpha^{h_1}(h_0^*)\cdot \varphi^{\text{KT}(h_0^*,V)}}\\
>&\frac{\sum_{h_0^*\in T_{m-2}} \Lambda_\alpha^{h_1}(h_0^*)\cdot \varphi^{\text{KT}(h_0^*,V)}\cdot \varphi^{\text{KT}(V,\text{Down}_a^1(V))}}{\sum_{h_0^*\in T_{m-2}} \Lambda_\alpha^{h_1}(h_0^*)\cdot \varphi^{\text{KT}(h_0^*,V)}}\\
=&\varphi=\frac{\pi_{h_1}(\text{Down}_a^1(V))}{\pi_{h_1}(V)}
\end{aligned}$$

The strict inequality holds because of (1) triangle inequality for Kentall-Tau distance, and (2) for any ranking V where the top-ranked alternative in h_0^* is ranked right below a, we have $\text{KT}(h_0^*,V)+\text{KT}(V,\text{Down}_a^1(V)) > \text{KT}(h_0^*,\text{Down}_a^1(V))$, and (3) for all $h_0^*\in T_{m-2}$, $\Lambda_\alpha^{h_1}(h_0^*)>0$.

It follows from the strict inequality that

$$\begin{aligned}
\text{Ratio}(V)=&\frac{\pi_{h_1}(V)}{\sum_{h_0^*\in T_{m-2}} \Lambda_\alpha^{h_1}(h_0^*)\cdot \pi_{h_0^*}(V)}\\
>&\frac{\pi_{h_1}(\text{Down}_a^1(V))}{\sum_{h_0^*\in T_{m-2}} \Lambda_\alpha^{h_1}(h_0^*)\cdot \pi_{h_0^*}(\text{Down}_a^1(V))}\\
=&\text{Ratio}(\text{Down}_a^1(V))
\end{aligned}$$

Moreover, for any $V,V'\in T_{s_\alpha}$ we have $\text{Ratio}(V)=\text{Ratio}(V')$ by verifying $\text{LP}_{s_\alpha}^{h_1}$. Therefore, for any $V\in T_i$ with $i<s_\alpha$, we can move up the position of a one by one until we reach the $(m-s_\alpha)$-th position. Let $V^*\in T_{s_\alpha}$ denote this ranking. It follows that $\text{Ratio}(V)<\text{Ratio}(V^*)$. Similarly for any $V'\in T_i$ with $i>s_\alpha$ we have $\text{Ratio}(V')>\text{Ratio}(V^*)$ for any $V^*\in T_{s_\alpha}$. This means that for any V where a is ranked above the $(m-s_\alpha)$-th position, we have $\text{LR}_\alpha(V)=1$; for any V where a is ranked below the $(m-s_\alpha)$-th position, we have $\text{LR}_\alpha(V)=0$; for any V where a is ranked at the $(m-s_\alpha)$-th position, we have that $\text{LR}_\alpha(V)$ is the same and is between 0 and 1. It follows that $\text{LR}_\alpha=\bar{f}_{\alpha,a}$.

Step 3. Due to the symmetry $f_{\alpha,a}$ among alternatives in $\mathcal{A}-\{a\}$, for any $i\le m-2$ and any $h_0,h_0'\in T_i$, we have $\text{Size}(\bar{f}_{\alpha,a},h_0)=\text{Size}(\bar{f}_{\alpha,a},h_0')$. Therefore, condition (i) in Lemma 2 is satisfied. Choose arbitrary $h_0^{m-2}\in T_{m-2}$. For any $i\le m-3$, let $h_0^i\in T_i$ denote the ranking obtained from h_0^{i+1} by moving a down for one position. To verify condition (ii) in Lemma 2, it suffices to prove that for any $i\le m-3$ and any $K\in\mathbb{N}$, we have

$$\begin{aligned}
&\pi_{h_0^{m-2}}(\{V:\text{Borda}_a(V)\ge K\})\\
\ge\ &\pi_{h_0^i}(\{V:\text{Borda}_a(V)\ge K\})
\end{aligned} \tag{2}$$

We will prove a slightly stronger lemma.

Lemma 8 *Under Mallows' model, for any m, any φ, any $W\in\mathcal{L}(\mathcal{A})$, any $b,c\in\mathcal{A}$ such that $b\succ_W c$, and any K, we have $\pi_W(\{V:\textit{Borda}_b(V)\ge K\})\ge\pi_W(\{V:\textit{Borda}_c(V)\ge K\})$.*

Proof: The proof is similar to the proof of Lemma 5. It suffices to prove the lemma for the case where b and c are adjacent in W. Let $\mathcal{P}=\{V\in\mathcal{L}(\mathcal{A}):\text{Borda}_b(V)\ge K\}$ and $\mathcal{P}'=\{V\in\mathcal{L}(\mathcal{A}):\text{Borda}_c(V)\ge K\}$. It follows that $\mathcal{P}\cap\mathcal{P}'$ is the set of rankings where both b and c are ranked within top $m-K$ positions; $\mathcal{P}-\mathcal{P}'$ is the set of rankings where b is ranked within top $m-K$ positions but c is not; and $\mathcal{P}'-\mathcal{P}$ is the set of rankings where c is ranked within top $m-K$ positions but b is not. We let $\mathcal{M}$ be a permutation that switches b and c. It is not hard to check that $\mathcal{M}$ is a bijection between $(\mathcal{P}-\mathcal{P}')$ and $(\mathcal{P}'-\mathcal{P})$, and because b and c are adjacent in W, for any $V\in\mathcal{P}$, we have $\text{KT}(M(V),W)=\text{KT}(V,W)+1$, which means that $\pi_W(V)=\pi(M(V))/\varphi$. Therefore, we have

$$
\begin{aligned}
&\pi_W(\{V : \text{Borda}_b(V) \geq K\}) - \pi_W(\{V : \text{Borda}_c(V) \geq K\}) \\
=&\pi_W(\mathcal{P}) - \pi_W(\mathcal{P}') = \pi_W(\mathcal{P} - \mathcal{P}') - \pi_W(\mathcal{P}' - \mathcal{P}) \\
=&\pi_W(\mathcal{P} - \mathcal{P}') - \pi_W(M(\mathcal{P} - \mathcal{P}')) \\
=&(\frac{1}{\varphi} - 1)\pi_W(\mathcal{P} - \mathcal{P}') \geq 0
\end{aligned}
$$

This proves the lemma. □

Let W be an arbitrary ranking and let M_i denote a permutation such that $M_i(h_0^i) = W$. We have $\pi_{h_0^i}(\{V : \text{Borda}_a(V) \geq K\}) = \pi_{M_i(h_0^i)}(\{V : \text{Borda}_{M_i(a)}(V) \geq K\})$. We note that $M_i(a)$ is the alternative that is ranked at the $(m-i)$-th position in W. Inequality (2) follows after applying Lemma 8. This means that condition (ii) in Lemma 2 is also satisfied. Therefore, by Lemma 2, $\bar{f}_{\alpha,a}$ is a level-α most powerful test for H_0 vs. h_1. Since $\bar{f}_{\alpha,a}$ does not depend on h_1, it is a level-α UMP test for H_0 vs. H_1. □

Lemma 6. For any $\mathcal{M}_X$ and $\mathcal{M}_Y$, suppose Λ_X is a least favorable distribution for composite vs. simple test ($H_{0,X}$ vs. x_1) under $\mathcal{M}_X$. Given $y_1 \in \Theta_Y$, let Λ^* be the distribution over $H_{0,X} \times \Theta_Y$ where for all $x \in H_{0,X}$, $\Lambda^*(x, y_1) = \Lambda_X(x)$. Then Λ^* is a least favorable distribution for $H_{0,X} \times \Theta_Y$ vs. (x_1, y_1) under $\mathcal{M}_X \otimes \mathcal{M}_Y$.

Proof: Let $x_0^1, \ldots, x_0^K \in \Theta_X$ denote the support of Λ_X. The theorem is proved by applying Lemma 2. For any $0 < \alpha < 1$ and any $P = (P_X, P_Y) \in \mathcal{S}_X \times \mathcal{S}_Y$, we have the following calculation. In this proof Ratio stands for $\text{Ratio}_{\Lambda^*,(x_1,y_1)}$ and LR_α stands for $\text{LR}_{\alpha,\Lambda^*,(x_1,y_1)}$.

$$
\begin{aligned}
\text{Ratio}(P_X, P_Y) =& \frac{\pi_{x_1,y_1}(P)}{\sum_{k=1}^K \Lambda^*(x_0^k, y_1)\pi_{(x_0^k,y_1)}(P)} \\
=& \frac{\pi_{x_1}(P_X) \cdot \pi_{y_1}(P_Y)}{\sum_{k=1}^K \Lambda^*(x_0^k, y_1)\pi_{x_0^k}(P_X) \cdot \pi_{y_1}(P_Y)} \\
=& \frac{\pi_{x_1}(P_X)}{\sum_{k=1}^K \Lambda(x_0^k)\pi^k_{x_0}(P_X)} = \text{Ratio}_{\Lambda,x_1}(P_X)
\end{aligned}
$$

It follows that for any pair of samples $(P_X, P_Y), (P'_X, P'_Y) \in \mathcal{S}_X \times \mathcal{S}_Y$, $\text{Ratio}(P_X, P_Y) \geq \text{Ratio}(P'_X, P'_Y)$ if and only if $\text{Ratio}_{\Lambda,x}(P_X) \geq \text{Ratio}_{\Lambda,x}(P'_X)$. This means that for any (P_X, P_Y), $\text{LR}_\alpha(P_X, P_Y) = \text{LR}_{\alpha,\Lambda,x_1}(P_X)$. Therefore, for any $x_0 \in H_{0,X}$, we have

$$
\begin{aligned}
&\text{Size}(\text{LR}_\alpha, (x_0, y_1)) \\
=& \sum_{(P_X,P_Y)\in\mathcal{S}_X\times\mathcal{S}_Y} \pi_{x_0}(P_X)\pi_{y_1}(P_Y)\text{LR}_\alpha(P_X, P_Y) \\
=& \sum_{(P_X,P_Y)\in\mathcal{S}_X\times\mathcal{S}_Y} \pi_{x_0}(P_X)\pi_{y_1}(P_Y)\text{LR}_{\alpha,\Lambda,x_1}(P_X) \\
=& \sum_{P_X\in\mathcal{S}_X} \pi_{x_0}(P_X)\text{LR}_{\alpha,\Lambda,x_1}(P_X) \\
=&\text{Size}(\text{LR}_{\alpha,\Lambda,x_1}, x_0)
\end{aligned}
$$

Therefore, by Lemma 2, for any $(x_0^*, y_1) \in \text{Spt}(\Lambda^*)$, we have $\text{Size}(\text{LR}_\alpha, (x_0, y_1)) = \text{Size}(\text{LR}_{\alpha,\Lambda,x_1}, x_0) = \alpha$ because $x_0^* \in \text{Spt}(\Lambda)$; for any $(x_0, y) \in H_{0,X} \times \Theta_Y$, we have $\text{Size}(\text{LR}_\alpha, (x_0, y)) = \text{Size}(\text{LR}_{\alpha,\Lambda,x_1}, x_0) \leq \alpha$. This means that the two conditions in Lemma 2 are satisfies, which proves the theorem. □

Lemma 7. *For any model $\mathcal{M}_X$ and any $t \in \mathbb{N}$, suppose Λ is a uniformly least favorable distribution for composite vs. simple test (H_0 vs. h_1) under $\mathcal{M}_X$. Then Ext(Λ, h_1, t) is a uniformly least favorable distribution for Ext(H_0, h_1, t) vs. $\vec{h}_1$ in $(\mathcal{M}_X)^t$.*

Proof: Again the proof is done by applying Lemma 2. We first prove a claim that characterizes samples whose likelihood ratio is no more than a given threshold. To this end, it is convenient to use the inverse of the likelihood ratio. To simplify notation, in this proof we let $\Lambda^* = \text{Ext}(\Lambda, h_1, t)$, let $H_0^* = \text{Ext}(H_0, h_1, t)$, let $\text{LR}_\alpha = \text{LR}_{\alpha,\Lambda^*,\vec{h}_1}$, $\text{Ratio} = \text{Ratio}_{\Lambda^*,\vec{h}_1}$.

Claim 1 *For any k_α and any $\vec{x} \in \mathcal{S}^t$, $\sum_{j=1}^t \mathit{Ratio}_{\Lambda,h_1}^{-1}(x_j) = t \cdot \mathit{Ratio}^{-1}(\vec{x})$.*

Proof: we have $\text{Ratio}^{-1}(\vec{x}) = \frac{1}{t} \cdot \frac{\sum_{j=1}^t \sum_{h_0 \in H_0} \Lambda(h_0) \cdot \pi_{(h_0,[\vec{h}_1]_{-j})}(\vec{x})}{\pi_{\vec{h}_1}(\vec{x})}$

$= \frac{1}{t} \cdot \frac{\sum_{j=1}^t \sum_{h_0 \in H_0} \Lambda(h_0) \cdot \pi_{h_0}(x_j) \cdot \pi_{[\vec{h}_1]_{-j}}(x_j)}{\pi_{h_1}(x_j) \cdot \pi_{[\vec{h}_1]_{-j}}(x_j)}$

$= \frac{1}{t} \sum_{j=1}^t \text{Ratio}_{\Lambda,h_1}^{-1}(x_j)$ □

The next lemma proves the following: For any $\vec{z} \in H_0^*$ and any $j \le t$, suppose the j-th component is not in $\text{Spt}(\Lambda) \cup \{h_1\}$. If we fix all components except j-th in $\vec{z}$ and change the j-th component to $h_0^* \in \text{Spt}(\Lambda)$, then the size of LR_α will increase. If we further change the j-th component to h_1, then the size of LR_α will further increase.

Lemma 9 *For any $0 \le \alpha \le 1$, any $j \le t$, any $\vec{z}_{-j} \in \Theta^{t-1}$, any $h_0 \in H_0$, and any $h_0^* \in \mathit{Spt}(\Lambda)$, we have $\mathit{Size}(\mathit{LR}_\alpha, (h_0, \vec{z}_{-j})) \le \mathit{Size}(\mathit{LR}_\alpha, (h_0^*, \vec{z}_{-j})) \le \mathit{Size}(\mathit{LR}_\alpha, (h_1, \vec{z}_{-j}))$.*

Proof: For any $\vec{z}_{-j} \in \Theta^{n-1}$, we have

$$\begin{aligned}&\text{Size}(\text{LR}_\alpha, (h_0, \vec{z}_{-j})) = \pi_{(h_0,\vec{z}_{-j})}(\{\vec{x} \in \mathcal{S}^t : \text{Ratio}(\vec{x}) > k_\alpha^*\})\\ &+ \gamma_\alpha^* \pi_{(h_0,\vec{z}_{-j})}(\{\vec{x} \in \mathcal{S}^t : \text{Ratio}(\vec{x}) = k_\alpha^*\})\end{aligned}$$

For any $\vec{x}$, we let $\text{Sum}(\vec{x}) = \sum_{l=1}^t \text{Ratio}_{\Lambda,h_1}^{-1}(x_l)$ and for any $j \le t$, we let $\text{Sum}(\vec{x}_{-j}) = \sum_{l \ne j} \text{Ratio}_{\Lambda,h_1}^{-1}(x_l)$. By Claim 1, we have

$$\begin{aligned}&\pi_{(h_0,\vec{z}_{-j})}(\{\vec{x} \in \mathcal{S}^t : \text{Ratio}(\vec{x}) > k_\alpha^*\})\\ =&\pi_{(h_0,\vec{z}_{-j})}(\{\vec{x} \in \mathcal{S}^t : \text{Sum}(\vec{x}) < t/k_\alpha^*\})\\ =&\pi_{(h_0,\vec{z}_{-j})}(\{\vec{x} \in \mathcal{S}^t : \text{Sum}(\vec{x}_{-j}) + \text{Ratio}_{\Lambda,h_1}^{-1}(x_j) < t/k_\alpha^*\})\\ =&\int_0^{t/k_\alpha^*} \sum\nolimits_{\vec{x}_{-j} \in \mathcal{S}^{t-1}:\text{Sum}(\vec{x}_{-j})=p}\\ &\sum\nolimits_{x_j:\text{Ratio}_{\Lambda,h_1}^{-1}(x_j) < t/k_\alpha^* - p} \pi_{(h_0,\vec{z}_{-j})}(\vec{x}) dp\\ =&\int_0^{t/k_\alpha^*} \pi_{\vec{z}_{-j}}(\{\vec{x}_{-j} \in \mathcal{S}^{t-1} : \text{Sum}(\vec{x}_{-j}) = p\})\\ &\cdot \pi_{h_0}(\{x_j : \text{Ratio}_{\Lambda,h_1}^{-1}(x_j) < t/k_\alpha^* - p\}) dp\\ =&\int_0^{t/k_\alpha^*} Q(\vec{z}_{-j}, p) \cdot \pi_{h_0}(\{x_j : \text{Ratio}_{\Lambda,h_1}^{-1}(x_j) < t/k_\alpha^* - p\}) dp\end{aligned}$$

where $Q(\vec{z}_{-j}, p) = \pi_{\vec{z}_{-j}}(\{\vec{x}_{-j} \in \mathcal{S}^{t-1} : \text{Sum}(\vec{x}_{-j}) = p\})$. Given p and γ_α^*, let α' denote the size of the likelihood ratio test $\text{LR}_{\alpha',\Lambda,h_1}$, where the threshold $k_{\alpha'}$ is $1/(t/k_\alpha^* - p)$ and $\gamma_{\alpha'} = \gamma_\alpha^*$. We have

$$\text{Size}(\text{LR}_\alpha, (h_0, \vec{z}_{-j})) = \int_0^{t/k_\alpha^*} Q(\vec{z}_{-j}, p) \cdot \text{Size}(\text{LR}_{\alpha',\Lambda,h_1}, h_0) dp \tag{3}$$

We note that in Equation (3), α' is a function of t, p, k_α^*, and γ_α^*. Because Λ is a uniformly least favorable distribution, it follows from Lemma 2 that for any $h_0^* \in \text{Spt}(\Lambda)$ and any $h_0 \in (H_0 - \text{Spt}(\Lambda))$, we have

$$\text{Size}(\text{LR}_{\alpha',\Lambda,h_1}, h_0) \le \alpha' \le \text{Size}(\text{LR}_{\alpha',\Lambda,h_1}, h_0^*)$$

Then by Equation (3), for any $h_0 \in (H_0 - \text{Spt}(\Lambda))$ and any $h_0^* \in \text{Spt}(\Lambda)$, we have

$$\begin{aligned}
&\text{Size}(\text{LR}_\alpha, (h_0, \vec{z}_{-j})) \\
=&\int_0^{t/k_\alpha^*} Q(\vec{z}_{-j}, p) \cdot \text{Size}(\text{LR}_{\alpha',\Lambda,h_1}, h_0) dp \\
\leq&\int_0^{t/k_\alpha^*} Q(\vec{z}_{-j}, p) \cdot \text{Size}(\text{LR}_{\alpha',\Lambda,h_1}, h_0^*) dp \\
=&\text{Size}(\text{LR}_\alpha, (h_0^*, \vec{z}_{-j}))
\end{aligned}$$

To prove the last inequality in the lemma, we prove a claim that holds for any least favorable distribution and the corresponding likelihood ratio test. The Size$(\cdot)$ function in the claim is extended to $h_1 \in H_1$ in the natural way.

Claim 2 *For any model, any composite vs. simple test (H_0 vs. h_1), suppose Λ is a level-η least favorable distribution. Then we have Size$(LR_\eta, h_1) \geq \eta =$ Size(LR_η, h_0^Λ).* [4]

Proof: For the sake of contradiction suppose this is not true, that is, for any $h_0^* \in \text{Spt}(\Lambda)$ we have $\text{Size}(\text{LR}_\eta, h_1) < \eta = \text{Size}(\text{LR}_\eta, h_0^*)$. It follows that $k_\eta \leq 1$, otherwise we have

$$\begin{aligned}
&\text{Size}(\text{LR}_\eta, h_1) \\
=&\sum_{P\in\mathcal{S}:\text{Ratio}(P)>k_\eta} \pi_{h_1}(P) + \gamma_\eta \sum_{P\in\mathcal{S}:\text{Ratio}(P)=k_\eta} \pi_{h_1}(P) \\
\geq&\sum_{P\in\mathcal{S}:\text{Ratio}(P)>k_\eta} \pi_\Lambda(P)\cdot k_\eta + \gamma_\eta \sum_{P\in\mathcal{S}:\text{Ratio}(P)=k_\eta} \pi_\Lambda(P)\cdot k_\eta \\
>&\sum_{P\in\mathcal{S}:\text{Ratio}(P)>k_\eta} \pi_\Lambda(P) + \gamma_\eta \sum_{P\in\mathcal{S}:\text{Ratio}(P)=k_\eta} \pi_\Lambda(P) = \eta,
\end{aligned}$$

which is a contradiction. Therefore, we have

$$\begin{aligned}
&1 \\
=&\text{Size}(\text{LR}_\eta, h_1) + \sum\nolimits_{P\in\mathcal{S}:\text{Ratio}(P)<k_\eta} \pi_{h_1}(P) \\
&+ (1-\gamma_\eta) \sum\nolimits_{P\in\mathcal{S}:\text{Ratio}(P)=k_\eta} \pi_{h_1}(P) \\
<&\eta + \sum\nolimits_{P\in\mathcal{S}:\text{Ratio}(P)<k_\eta} \pi_\Lambda(P)\cdot k_\eta \\
&+ (1-\gamma_\eta) \sum\nolimits_{P\in\mathcal{S}:\text{Ratio}(P)=k_\eta} \pi_\Lambda(P)\cdot k_\eta \\
\leq&\eta + k_\eta(1 - \text{Size}(\text{LR}_\eta, h_0^\Lambda)) \leq 1,
\end{aligned}$$

which is a contradiction. □

Applying Claim 2 to $\text{LR}_{\alpha',\Lambda,h_1}$, we have

$$\begin{aligned}
&\text{Size}(\text{LR}_\alpha, (h_0^*, \vec{z}_{-j})) \\
=&\int_0^{t/k_\alpha^*} Q(\vec{z}_{-j}, p) \cdot \text{Size}(\text{LR}_{\alpha',\Lambda,h_1}, h_0^*) dp \\
\leq&\int_0^{t/k_\alpha^*} Q(\vec{z}_{-j}, p) \cdot \text{Size}(\text{LR}_{\alpha',\Lambda,h_1}, h_1) dp \\
=&\text{Size}(\text{LR}_\alpha, (h_1, \vec{z}_{-j}))
\end{aligned}$$

[4] We recall that h_0^Λ is the combined H_0 by Λ.

This finishes the proof of Lemma 9. □

It follows from Lemma 9 that for any $j \leq t$ and any $h_0^* \in \text{Spt}(\Lambda)$, we have that $\text{Size}(\text{LR}_\alpha, (h_0^*, [\vec{h}_1]_{-j}))$ is the same. Due to symmetry, for any $\vec{h}_0^* \in H_0^*$, $\text{Size}(\text{LR}_\alpha, h_0^*)$ is the same and is therefore equivalent to α. This verifies condition (i) in Lemma 2.

Condition (ii) in Lemma 2 is verified by recursively applying Lemma 9. Given any $\vec{h}_0 \in H_0^* - \text{Spt}(\Lambda^*)$, there must exist $j \leq t$ such that $[\vec{h}_0]_j \neq h_1$. We then change $[\vec{h}_0]_j$ to an arbitrary $h_0^* \in \text{Spt}(\Lambda)$, then change the other components of $\vec{h}_0$ to h_1 one by one. Each time we make the change the size of LR_α does not decrease according to Lemma 9. At the end of the process we obtain $(h_0^*, [\vec{h}_1]_j) \in \text{Spt}(\Lambda^*)$, at which the size of LR_α is α. The theorem follows after applying Lemma 2. □

We now define a test $\bar{f}_{\alpha,a}$ for $H_0 = (\mathcal{L}(\mathcal{A}) - H_1)$ vs. $H_1 = L_{a \succ \text{others}}$ and prove that if a UMP test exists, then $\bar{f}_{\alpha,a}$ must also be a UMP test. For any $V \in \mathcal{L}(\mathcal{A})$ and any alternative $a \in \mathcal{A}$, we let $\text{Borda}_a(V)$ denote the Borda score of a in V. That is, $\text{Borda}_a(V)$ is the number of alternatives that are ranked below a in V. For any $V \in \mathcal{L}(\mathcal{A})$, we let $\bar{f}_{\alpha,a}(V) = \begin{cases} 1 & \text{if Borda}_a(V) > K_\alpha \\ 0 & \text{if Borda}_a(V) < K_\alpha \\ \Gamma_\alpha & \text{if Borda}_a(V) = K_\alpha \end{cases}$, where K_α and Γ_α are chosen so that the size of $\bar{f}_{\alpha,a}$ is α. In other words, $\bar{f}_{\alpha,a}$ calculates the Borda score of a in the input profile, and if it is larger than a threshold K_α then H_0 is rejected. It is not hard to see that $\bar{f}_{\alpha,a}$ equals to $f_{\alpha',a}$ with a possibly different level α' (defined in Theorem 3).

Lemma 10 *If there exists a level-α UMP test for $H_0 = (\mathcal{L}(\mathcal{A}) - H_1)$ vs. $H_1 = L_{a \succ \text{others}}$, then $\bar{f}_{\alpha,a}$ is also a level-α UMP test.*

Proof: Let f_α denote a level-α UMP test. For any permutation M over $\mathcal{A} - \{a\}$, we let $M(f_\alpha)$ denote the test such that for any $V \in \mathcal{L}(\mathcal{A})$, $M(f_\alpha)(V) = f_\alpha(M(V))$. Because the Kendall-Tau distance is invariant to permutations, we have that for any $h_0 \in H_0$, $\text{Size}(f_\alpha, h_0) = \text{Size}(M(f_\alpha), M(h_0))$, and for any $h_1 \in H_1$, $\text{Power}(f_\alpha, h_1) = \text{Power}(M(f_\alpha), M(h_1))$. Therefore $\text{Size}(M(f_\alpha)) = \alpha$. Also because the multi-set of $\{\text{Power}(f_\alpha, h_1) : h_1 \in H_1\}$ is the same as the multi-set $\{\text{Power}(M(f_\alpha), h_1) : h_1 \in H_1\}$, for all $h_1 \in H_1$, we must have $\text{Power}(f_\alpha, h_1) = \text{Power}(M(f_\alpha), h_1)$, otherwise there exists $h_1 \in H_1$ such that $\text{Power}(f_\alpha, h_1) < \text{Power}(M(f_\alpha), h_1)$, which contradicts the assumption that f_α is UMP.

It follows that for any permutation M over $\mathcal{A} - \{a\}$, $M(f_\alpha)$ is also UMP. Therefore, $\bar{f}_\alpha = \frac{1}{(m-1)!} \sum_M M(f_\alpha)$ is also UMP. We note that for any V, V' where a has the same Borda score, there exists a permutation M over $\mathcal{A} - \{a\}$ so that $M(V) = V'$. This means that $\bar{f}_\alpha(V) = \bar{f}_\alpha(V')$.

We now prove that $\bar{f}_\alpha$ must be $\bar{f}_{\alpha,a}$ as in the statement of the Lemma. More precisely, we will prove that for any V, V' such that $\text{Borda}_a(V) > \text{Borda}_a(V')$, if $\bar{f}_\alpha(V') > 0$ then $\bar{f}_\alpha(V) = 1$. Suppose for the sake of contradiction that this is not true, and there exist V, V' such that $s_1 = \text{Borda}_a(V) > \text{Borda}_a(V') = s_2$, $\bar{f}_\alpha(V') > 0$, and $\bar{f}_\alpha(V) < 1$. For any $s \leq m-1$, we let T_s denote the set of rankings where the Borda score of a is s. That is, $T_s = \{V \in \mathcal{L}(\mathcal{A}) : \text{Borda}_a(V) = s\}$. We will prove that for any $s_1 > s_2$, T_{s_1} as a whole is more "cost effective" than T_{s_2} as a whole for any $h_0 \in H_0$ against any $h_1 \in H_1$. More precisely, we will prove that $\text{Ratio}_{h_0,h_1}(T_{s_1}) > \text{Ratio}_{h_0,h_1}(T_{s_2})$.

For any $s \leq m-2$ and any $h_0 \in T_s$, let h_1 denote the ranking in $T_{m-1} = H_1$ that is obtained from θ by raising a to the top position. For any $V_{s_1} \in T_{s_1}$, we let $\text{Down}_a^{s_1-s_2}(V_{s_1}) \in T_{s_2}$ denote the ranking that is obtained from V_{s_1} by moving a down for $s_1 - s_2$ positions, that is, from the $(m - s_1)$-th position to the $(m - s_2)$-th position. We have

$$\begin{aligned}
&\frac{\pi_{h_0}(T_{s_2})}{\pi_{h_0}(T_{s_1})}\\
=&\frac{\sum_{V \in T_{s_2}} \pi_{h_0}(V)}{\sum_{V \in T_{s_1}} \pi_{h_0}(V)} = \frac{\sum_{V \in T_{s_1}} \pi_{h_0}(\text{Down}_a^{s_1-s_2}(V))}{\sum_{V \in T_{s_1}} \pi_{h_0}(V)}\\
=&\frac{\sum_{V \in T_{s_1}} \varphi^{\text{KT}(h_0, \text{Down}_a^{s_1-s_2}(V))}}{\sum_{V \in T_{s_1}} \varphi^{\text{KT}(h_0, V)}}\\
>&\frac{\sum_{V \in T_{s_1}} \varphi^{\text{KT}(h_0, V)} \cdot \varphi^{\text{KT}(V, \text{Down}_a^{s_1-s_2}(V))}}{\sum_{V \in T_{s_1}} \varphi^{\text{KT}(h_0, V)}}\\
=&\varphi^{s_1-s_2} = \frac{\pi_{h_1}(T_{s_2})}{\pi_{h_1}(T_{s_1})}
\end{aligned}$$

The inequality is due to triangle inequality for Kendall-Tau distance. It is strict because for any $V \in T_{s_1}$ where the top-ranked alternative in h_0 is ranked between the $(m - s_1)$-th and $(m - s_2)$-th position, $\text{KT}(h_0, \text{Down}_a^{s_1-s_2}(V)) < \text{KT}(h_0, V) + \text{KT}(V, \text{Down}_a^{s_1-s_2}(V))$. Therefore, $\frac{\pi_{h_0}(T_{s_2})}{\pi_{h_0}(T_{s_1})} > \frac{\pi_{h_1}(T_{s_2})}{\pi_{h_1}(T_{s_1})}$, which means that $\text{Ratio}_{h_0,h_1}(T_{s_1}) = \frac{\pi_{h_1}(T_{s_1})}{\pi_{h_0}(T_{s_1})} > \frac{\pi_{h_1}(T_{s_2})}{\pi_{h_0}(T_{s_2})} = \text{Ratio}_{h_0,h_1}(T_{s_2})$.

Therefore, we can find sufficiently small $\epsilon, \delta > 0$, and replace ϵT_{s_2} by δT_{s_1} without changing the size. This will increase the power of $\bar{f}_\alpha$ because T_{s_1} is strictly more cost effective than T_{s_2}, which contradicts the assumption that $\bar{f}_\alpha$ is a UMP test. Therefore, $\bar{f}_\alpha = \bar{f}_{\alpha,a}$, which proves the lemma. □