Efficient Bayesian Inference for a Gaussian Process Density Model: Supplementary Material

A THE CONDITIONAL POSTERIOR POINT PROCESS

Here we prove that the conditional posterior point process in Equation (13) again is a Poisson process using Campbell's theorem [1, chap. 3]. For an arbitrary function $h(\cdot,\cdot)$ we set $H \doteq \sum_{(\boldsymbol{x},\omega)\in\Pi} h(\boldsymbol{x},\omega)$. We calculate the characteristic functional

$$
\begin{aligned}
&\mathbb{E}_{\phi_\lambda}\left[e^H \middle| g,\lambda\right] = \\
&\frac{\mathbb{E}_{\phi_\lambda}\left[\prod_{(\omega,\boldsymbol{x})\in\Pi} e^{f(\omega,-g(\boldsymbol{x}))+h(\boldsymbol{x},\omega)} \middle| g,\lambda\right]}{\exp\left(\int_{\mathcal{X}\times\mathbb{R}^+}\left(e^{f(\omega,-g(\boldsymbol{x}))}-1\right)\phi_\lambda(\boldsymbol{x},\omega)d\omega d\boldsymbol{x}\right)} = \\
&\frac{\exp\left\{\int_{\mathcal{X}\times\mathbb{R}^+}\left(e^{f(\omega,-g(\boldsymbol{x}))+h(\boldsymbol{x},\omega)}-1\right)\phi_\lambda(\boldsymbol{x},\omega)d\omega d\boldsymbol{x}\right\}}{\exp\left(\int_{\mathcal{X}\times\mathbb{R}^+}\left(e^{f(\omega,-g(\boldsymbol{x}))}-1\right)\phi_\lambda(\boldsymbol{x},\omega)d\omega d\boldsymbol{x}\right)} = \\
&\exp\left\{\int_{\mathcal{X}\times\mathbb{R}^+}\left(e^{h(\boldsymbol{x},\omega)}-1\right)e^{f(\omega,-g)}\phi_\lambda(\boldsymbol{x},\omega)d\omega d\boldsymbol{x}\right\} = \\
&\exp\left\{\int_{\mathcal{X}\times\mathbb{R}^+}\left(e^{h(\boldsymbol{x},\omega)}-1\right)\Lambda(\boldsymbol{x},\omega)d\omega d\boldsymbol{x}\right\},
\end{aligned}
$$

where the last equality follows from the definition of $\phi_\lambda(\boldsymbol{x},\omega)$ and the tilted Pólya–Gamma density. Using the fact that a Poisson process is uniquely characterised by its generating function this shows that the conditional posterior $p(\Pi|g,\lambda)$ is a marked Poisson process.

B VARIATIONAL LOWER BOUND

The full variational lower bound is given by

$$
\begin{aligned}
\mathcal{L}(q) = &\sum_{n=1}^{N}\left\{\mathbb{E}_Q\left[\ln\lambda\right]+\ln\pi(\boldsymbol{x}_n)+\mathbb{E}_Q\left[f(\omega_n,g(\boldsymbol{x}_n))\right]-\ln\cosh\left(\frac{c_n}{2}\right)+\frac{c_n^2}{2}\mathbb{E}_Q\left[\omega_n\right]\right\} \\
&+\int_{\mathcal{X}}\int_{\mathbb{R}^+}\left\{\mathbb{E}_Q\left[\ln\lambda\right]+\mathbb{E}_Q\left[f(\omega,-g(\boldsymbol{x}))\right]-\ln\lambda_1-\ln\sigma(-c(\boldsymbol{x}))-\ln\cosh\left(\frac{c(\boldsymbol{x})}{2}\right)-\frac{c(\boldsymbol{x})^2}{2}\omega\right. \\
&\left.-\frac{c(\boldsymbol{x})-g_1(\boldsymbol{x})}{2}+1\right\}\Lambda_1(\boldsymbol{x},\omega)d\omega d\boldsymbol{x}-\mathbb{E}_Q\left[\lambda\right]+\mathbb{E}_Q\left[\ln\frac{p(\lambda)}{q(\lambda)}\right]+\mathbb{E}_Q\left[\ln\frac{p(\boldsymbol{g}_s)}{q(\boldsymbol{g}_s)}\right].
\end{aligned}
$$

References

[1] John Frank Charles Kingman. *Poisson processes*. Wiley Online Library, 1993.