Appendix

A Proof for Theorem 1

A.1 Notations

We start by defining some notations. For each time t, we define a random permutation $(\mathbf{a}_1^{*,t}, \ldots, \mathbf{a}_K^{*,t})$ of A^* based on $\mathbf{A}_t$ as follows: for any $k = 1, \ldots, K$, if $\mathbf{a}_k^t \in A^*$, then we set $\mathbf{a}_k^{*,t} = \mathbf{a}_k^t$. The remaining optimal items are positioned arbitrarily. Notice that under this random permutation, we have:

$$\bar{w}(\mathbf{a}_k^{*,t}) \geq \bar{w}(\mathbf{a}_k^t) \quad \text{and} \quad \mathbf{U}_t(\mathbf{a}_k^t) \geq \mathbf{U}_t(\mathbf{a}_k^{*,t}) \quad \forall k = 1, 2, \ldots, K$$

Moreover, we use $\mathcal{H}_t$ to denote the "history" (rigorously speaking, σ-algebra) by the end of time t. Then both $\mathbf{A}_t = (\mathbf{a}_1^t, \ldots, \mathbf{a}_K^t)$ and the permutation $(\mathbf{a}_1^{*,t}, \ldots, \mathbf{a}_K^{*,t})$ of A^* are $\mathcal{H}_{t-1}$-adaptive. In other words, they are conditionally deterministic at the beginning of time t. To simplify the notation, in this paper, we use $\mathbb{E}_t[\cdot]$ to denote $\mathbb{E}[\cdot|\mathcal{H}_{t-1}]$ when appropriate.

When appropriate, we also use $\langle \cdot, \cdot \rangle$ to denote the inner product of two vectors. Specifically, for two vectors u and v with the same dimension, we use $\langle u, v \rangle$ to denote $u^\intercal v$.

A.2 Regret Decomposition

We first prove the following technical lemma:

Lemma 1. *For any $B = (b_1, \ldots, b_K) \in \Re^K$ and $C = (c_1, \ldots, c_K) \in \Re^K$, we have*

$$\prod_{k=1}^K b_k - \prod_{k=1}^K c_k = \sum_{k=1}^K \left[\prod_{i=1}^{k-1} b_i\right] \times [b_k - c_k] \times \left[\prod_{j=k+1}^K c_j\right].$$

Proof. Notice that

$$\begin{aligned} &\sum_{k=1}^K \left[\prod_{i=1}^{k-1} b_i\right] \times [b_k - c_k] \times \left[\prod_{j=k+1}^K c_j\right] \\ =&\sum_{k=1}^K \left\{\left[\prod_{i=1}^{k} b_i\right] \times \left[\prod_{j=k+1}^K c_j\right] - \left[\prod_{i=1}^{k-1} b_i\right] \times \left[\prod_{j=k}^K c_j\right]\right\} \\ =&\prod_{k=1}^K b_k - \prod_{k=1}^K c_k. \end{aligned}$$

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Thus we have

$$\begin{aligned} R(\mathbf{A}_t, \mathbf{w}_t) =& f(A^*, \mathbf{w}_t) - f(\mathbf{A}_t, \mathbf{w}_t) \\ =& \prod_{k=1}^K (1 - \mathbf{w}_t(\mathbf{a}_k^t)) - \prod_{k=1}^K \left(1 - \mathbf{w}_t(\mathbf{a}_k^{*,t})\right) \\ \stackrel{(a)}{=}& \sum_{k=1}^K \left[\prod_{i=1}^{k-1} (1 - \mathbf{w}_t(\mathbf{a}_i^t))\right] \left[\mathbf{w}_t(\mathbf{a}_k^{*,t}) - \mathbf{w}_t(\mathbf{a}_k^t)\right] \left[\prod_{j=k+1}^K \left(1 - \mathbf{w}_t(\mathbf{a}_j^{*,t})\right)\right] \\ \stackrel{(b)}{\leq}& \sum_{k=1}^K \left[\prod_{i=1}^{k-1} (1 - \mathbf{w}_t(\mathbf{a}_i^t))\right] \left[\mathbf{w}_t(\mathbf{a}_k^{*,t}) - \mathbf{w}_t(\mathbf{a}_k^t)\right], \end{aligned} \tag{5}$$

where equality (a) is based on Lemma 1 and inequality (b) is based on the fact that $\prod_{j=k+1}^K \left(1 - \mathbf{w}_t(\mathbf{a}_j^{*,t})\right) \leq 1$. Recall that $\mathbf{A}^t$ and the permutation $(\mathbf{a}_1^{*,t}, \ldots, \mathbf{a}_K^{*,t})$ of A^* are deterministic conditioning on $\mathcal{H}_{t-1}$, and $\mathbf{a}_k^{*,t} \neq \mathbf{a}_i^t$ for all $i < k$, thus we have

$$\begin{aligned} \mathbb{E}_t[R(\mathbf{A}_t, \mathbf{w}_t)] \leq& \mathbb{E}_t\left[\sum_{k=1}^K \left[\prod_{i=1}^{k-1} (1 - \mathbf{w}_t(\mathbf{a}_i^t))\right] \left[\mathbf{w}_t(\mathbf{a}_k^{*,t}) - \mathbf{w}_t(\mathbf{a}_k^t)\right]\right] \\ =& \sum_{k=1}^K \mathbb{E}_t\left[\prod_{i=1}^{k-1} (1 - \mathbf{w}_t(\mathbf{a}_i^t))\right] \mathbb{E}_t\left[\mathbf{w}_t(\mathbf{a}_k^{*,t}) - \mathbf{w}_t(\mathbf{a}_k^t)\right] \\ =& \sum_{k=1}^K \mathbb{E}_t\left[\prod_{i=1}^{k-1} (1 - \mathbf{w}_t(\mathbf{a}_i^t))\right] \left[\bar{w}(\mathbf{a}_k^{*,t}) - \bar{w}(\mathbf{a}_k^t)\right]. \end{aligned}$$

For any $t \le n$ and any $e \in E$, we define event

$$\mathcal{G}_{t,k} = \left\{ \text{item } \mathbf{a}_k^t \text{ is examined in episode } t \right\},$$

notice that $\mathbb{1}\{\mathcal{G}_{t,k}\} = \prod_{i=1}^{k-1} (1 - \mathbf{w}_t(\mathbf{a}_i^t))$. Thus, we have

$$\mathbb{E}_t[\mathbf{R}_t] \le \sum_{k=1}^K \mathbb{E}_t[\mathbb{1}\{\mathcal{G}_{t,k}\}] \left[\bar{w}(\mathbf{a}_k^{*,t}) - \bar{w}(\mathbf{a}_k^t) \right].$$

Hence, from the tower property, we have

$$R(n) \le \mathbb{E}\left[\sum_{t=1}^n \sum_{k=1}^K \mathbb{1}\{\mathcal{G}_{t,k}\} \left[\bar{w}(\mathbf{a}_k^{*,t}) - \bar{w}(\mathbf{a}_k^t) \right] \right]. \tag{6}$$

We further define event $\mathcal{E}$ as

$$\mathcal{E} = \left\{ \left| \langle x_e, \bar{\theta}_{t-1} - \theta^* \rangle \right| \le c \sqrt{x_e^T M_{t-1}^{-1} x_e}, \, \forall e \in E, \, \forall t \le n \right\}, \tag{7}$$

and $\bar{\mathcal{E}}$ as the complement of $\mathcal{E}$. Then we have

$$\begin{aligned} R(n) &\overset{(a)}{\le} P(\mathcal{E}) \mathbb{E}\left[\sum_{t=1}^n \sum_{k=1}^K \mathbb{1}\{\mathcal{G}_{t,k}\} \left[\bar{w}(\mathbf{a}_k^{*,t}) - \bar{w}(\mathbf{a}_k^t) \right] \Big| \mathcal{E} \right] \\ &\quad + P(\bar{\mathcal{E}}) \mathbb{E}\left[\sum_{t=1}^n \sum_{k=1}^K \mathbb{1}\{\mathcal{G}_{t,k}\} \left[\bar{w}(\mathbf{a}_k^{*,t}) - \bar{w}(\mathbf{a}_k^t) \right] \Big| \bar{\mathcal{E}} \right] \\ &\overset{(b)}{\le} \mathbb{E}\left[\sum_{t=1}^n \sum_{k=1}^K \mathbb{1}\{\mathcal{G}_{t,k}\} \left[\bar{w}(\mathbf{a}_k^{*,t}) - \bar{w}(\mathbf{a}_k^t) \right] \Big| \mathcal{E} \right] + nKP(\bar{\mathcal{E}}), \end{aligned} \tag{8}$$

where inequality (a) is based on the law of total probability, and the inequality (b) is based on the naive bounds (1) $P(\mathcal{E}) \le 1$ and (2) $\mathbb{1}\{\mathcal{G}_{t,k}\} \left[\bar{w}(\mathbf{a}_k^{*,t}) - \bar{w}(\mathbf{a}_k^t) \right] \le 1$. Notice that from the definition of event $\mathcal{E}$, we have

$$\bar{w}(e) = \langle x_e, \theta^* \rangle \le \langle x_e, \bar{\theta}_{t-1} \rangle + c \sqrt{x_e^T M_{t-1}^{-1} x_e} \quad \forall e \in E, \, \forall t \le n$$

under event $\mathcal{E}$. Moreover, since $\bar{w}(e) \le 1$ by definition, we have $\bar{w}(e) \le \mathbf{U}_t(e)$ for all $e \in E$ and all $t \le n$ under event $\mathcal{E}$. Hence under event $\mathcal{E}$, we have

$$\bar{w}(\mathbf{a}_k^t) \le \bar{w}(\mathbf{a}_k^{*,t}) \le \mathbf{U}_t(\mathbf{a}_k^{*,t}) \le \mathbf{U}_t(\mathbf{a}_k^t) \le \langle x_{\mathbf{a}_k^t}, \bar{\theta}_{t-1} \rangle + c \sqrt{x_{\mathbf{a}_k^t}^T M_{t-1}^{-1} x_{\mathbf{a}_k^t}} \quad \forall t \le n.$$

Thus we have

$$\begin{aligned} \bar{w}(\mathbf{a}_k^{*,t}) - \bar{w}(\mathbf{a}_k^t) &\overset{(a)}{\le} \langle x_{\mathbf{a}_k^t}, \bar{\theta}_{t-1} - \theta^* \rangle + c \sqrt{x_{\mathbf{a}_k^t}^T M_{t-1}^{-1} x_{\mathbf{a}_k^t}} \\ &\overset{(b)}{\le} 2c \sqrt{x_{\mathbf{a}_k^t}^T M_{t-1}^{-1} x_{\mathbf{a}_k^t}}, \end{aligned}$$

where inequality (a) follows from the fact that $\bar{w}(\mathbf{a}_k^{*,t}) \le \langle x_{\mathbf{a}_k^t}, \bar{\theta}_{t-1} \rangle + c \sqrt{x_{\mathbf{a}_k^t}^T M_{t-1}^{-1} x_{\mathbf{a}_k^t}}$ and inequality (b) follows from the fact that $\langle x_{\mathbf{a}_k^t}, \bar{\theta}_{t-1} - \theta^* \rangle \le c \sqrt{x_{\mathbf{a}_k^t}^T M_{t-1}^{-1} x_{\mathbf{a}_k^t}}$ under event $\mathcal{E}$. Thus, we have

$$R(n) \le 2c \mathbb{E}\left[\sum_{t=1}^n \sum_{k=1}^K \mathbb{1}\{\mathcal{G}_{t,k}\} \sqrt{x_{\mathbf{a}_k^t}^T M_{t-1}^{-1} x_{\mathbf{a}_k^t}} \Big| \mathcal{E} \right] + nKP(\bar{\mathcal{E}}).$$

Define $\mathbf{K}_t = \min\{\mathbf{C}_t, K\}$, notice that

$$\sum_{k=1}^K \mathbb{1}\{\mathcal{G}_{t,k}\} \sqrt{x_{\mathbf{a}_k^t}^T M_{t-1}^{-1} x_{\mathbf{a}_k^t}} = \sum_{k=1}^{\mathbf{K}_t} \sqrt{x_{\mathbf{a}_k^t}^T M_{t-1}^{-1} x_{\mathbf{a}_k^t}}.$$

Thus, we have

$$R(n) \le 2c \mathbb{E}\left[\sum_{t=1}^n \sum_{k=1}^{\mathbf{K}_t} \sqrt{x_{\mathbf{a}_k^t}^T M_{t-1}^{-1} x_{\mathbf{a}_k^t}} \Big| \mathcal{E} \right] + nKP(\bar{\mathcal{E}}). \tag{9}$$

In the next two subsections, we will provide a *worst-case* bound on $\sum_{t=1}^n \sum_{k=1}^{\mathbf{K}_t} \sqrt{x_{\mathbf{a}_k^t}^T M_{t-1}^{-1} x_{\mathbf{a}_k^t}}$ and a bound on $P(\bar{\mathcal{E}})$.

A.3 Worst-Case Bound on $\sum_{t=1}^{n}\sum_{k=1}^{\mathbf{K}_t}\sqrt{x_{\mathbf{a}_k^t}^T M_{t-1}^{-1} x_{\mathbf{a}_k^t}}$

Lemma 2. $\sum_{t=1}^{n}\sum_{k=1}^{\mathbf{K}_t}\sqrt{x_{\mathbf{a}_k^t}^T M_{t-1}^{-1} x_{\mathbf{a}_k^t}} \leq K\sqrt{\frac{dn\log\left[1+\frac{nK}{d\sigma^2}\right]}{\log\left(1+\frac{1}{\sigma^2}\right)}}$.

Proof. To simplify the exposition, we define $z_{t,k} = \sqrt{x_{\mathbf{a}_k^t}^T M_{t-1}^{-1} x_{\mathbf{a}_k^t}}$ for all (t,k) s.t. $k \leq \mathbf{K}_t$. Recall that

$$M_t = M_{t-1} + \frac{1}{\sigma^2}\sum_{k=1}^{\mathbf{K}_t} x_{\mathbf{a}_k^t} x_{\mathbf{a}_k^t}^T$$

Thus, for all (t,k) s.t. $k \leq \mathbf{K}_t$, we have that

$$\begin{aligned}
\det\left[M_t\right] &\geq \det\left[M_{t-1} + \frac{1}{\sigma^2} x_{\mathbf{a}_k^t} x_{\mathbf{a}_k^t}^T\right] = \det\left[M_{t-1}^{\frac{1}{2}}\left(I + \frac{1}{\sigma^2} M_{t-1}^{-\frac{1}{2}} x_{\mathbf{a}_k^t} x_{\mathbf{a}_k^t}^T M_{t-1}^{-\frac{1}{2}}\right) M_{t-1}^{\frac{1}{2}}\right] \\
&= \det\left[M_{t-1}\right]\det\left[I + \frac{1}{\sigma^2} M_{t-1}^{-\frac{1}{2}} x_{\mathbf{a}_k^t} x_{\mathbf{a}_k^t}^T M_{t-1}^{-\frac{1}{2}}\right] \\
&= \det\left[M_{t-1}\right]\left(1 + \frac{1}{\sigma^2} x_{\mathbf{a}_k^t}^T M_{t-1}^{-1} x_{\mathbf{a}_k^t}\right) = \det\left[M_{t-1}\right]\left(1 + \frac{z_{t,k}^2}{\sigma^2}\right).
\end{aligned}$$

Thus, we have

$$\left(\det\left[M_t\right]\right)^{\mathbf{K}_t} \geq \left(\det\left[M_{t-1}\right]\right)^{\mathbf{K}_t}\prod_{k=1}^{\mathbf{K}_t}\left(1 + \frac{z_{t,k}^2}{\sigma^2}\right).$$

Since $\det\left[M_t\right] \geq \det\left[M_{t-1}\right]$ and $\mathbf{K}_t \leq K$, we have

$$\left(\det\left[M_t\right]\right)^{K} \geq \left(\det\left[M_{t-1}\right]\right)^{K}\prod_{k=1}^{\mathbf{K}_t}\left(1 + \frac{z_{t,k}^2}{\sigma^2}\right).$$

So we have

$$\left(\det\left[M_n\right]\right)^{K} \geq \left(\det\left[M_0\right]\right)^{K}\prod_{t=1}^{n}\prod_{k=1}^{\mathbf{K}_t}\left(1 + \frac{z_{t,k}^2}{\sigma^2}\right) = \prod_{t=1}^{n}\prod_{k=1}^{\mathbf{K}_t}\left(1 + \frac{z_{t,k}^2}{\sigma^2}\right),$$

since $M_0 = I$. On the other hand, we have that

$$\text{trace}\left(M_n\right) = \text{trace}\left(I + \frac{1}{\sigma^2}\sum_{t=1}^{n}\sum_{k=1}^{\mathbf{K}_t} x_{\mathbf{a}_k^t} x_{\mathbf{a}_k^t}^T\right) = d + \frac{1}{\sigma^2}\sum_{t=1}^{n}\sum_{k=1}^{\mathbf{K}_t}\|x_{\mathbf{a}_k^t}\|_2^2 \leq d + \frac{nK}{\sigma^2},$$

where the last inequality follows from the fact that $\|x_{\mathbf{a}_k^t}\|_2 \leq 1$ and $\mathbf{K}_t \leq K$. From the trace-determinant inequality, we have $\frac{1}{d}\text{trace}\left(M_n\right) \geq \left[\det(M_n)\right]^{\frac{1}{d}}$, thus we have

$$\left[1 + \frac{nK}{d\sigma^2}\right]^{dK} \geq \left[\frac{1}{d}\text{trace}\left(M_n\right)\right]^{dK} \geq \left[\det(M_n)\right]^{K} \geq \prod_{t=1}^{n}\prod_{k=1}^{\mathbf{K}_t}\left(1 + \frac{z_{t,k}^2}{\sigma^2}\right).$$

Taking the logarithm, we have

$$dK\log\left[1 + \frac{nK}{d\sigma^2}\right] \geq \sum_{t=1}^{n}\sum_{k=1}^{\mathbf{K}_t}\log\left(1 + \frac{z_{t,k}^2}{\sigma^2}\right). \tag{10}$$

Notice that $z_{t,k}^2 = x_{\mathbf{a}_k^t}^T M_{t-1}^{-1} x_{\mathbf{a}_k^t} \leq x_{\mathbf{a}_k^t}^T M_0^{-1} x_{\mathbf{a}_k^t} = \|x_{\mathbf{a}_k^t}\|_2^2 \leq 1$, thus we have $z_{t,k}^2 \leq \frac{\log\left(1+\frac{z_{t,k}^2}{\sigma^2}\right)}{\log\left(1+\frac{1}{\sigma^2}\right)}$.[5] Hence we have

$$\sum_{t=1}^{n}\sum_{k=1}^{\mathbf{K}_t} z_{t,k}^2 \leq \frac{1}{\log\left(1+\frac{1}{\sigma^2}\right)}\sum_{t=1}^{n}\sum_{k=1}^{\mathbf{K}_t}\log\left(1 + \frac{z_{t,k}^2}{\sigma^2}\right) \leq \frac{dK\log\left[1 + \frac{nK}{d\sigma^2}\right]}{\log\left(1+\frac{1}{\sigma^2}\right)}.$$

[5]Notice that for any $y \in [0,1]$, we have $y \leq \frac{\log\left(1+\frac{y}{\sigma^2}\right)}{\log\left(1+\frac{1}{\sigma^2}\right)} = h(y)$. To see it, notice that $h(y)$ is a strictly concave function, and $h(0) = 0$ and $h(1) = 1$.

Finally, from Cauchy-Schwarz inequality, we have that

$$\sum_{t=1}^{n}\sum_{k=1}^{\mathbf{K}_t} z_{t,k} \leq \sqrt{nK}\sqrt{\sum_{t=1}^{n}\sum_{k=1}^{\mathbf{K}_t} z_{t,k}^2} \leq K\sqrt{\frac{dn\log\left[1+\frac{nK}{d\sigma^2}\right]}{\log\left(1+\frac{1}{\sigma^2}\right)}}.$$

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A.4 Bound on $P(\bar{\mathcal{E}})$

Lemma 3. *For any $\sigma > 0$, any $\delta \in (0,1)$, and any*

$$c \geq \frac{1}{\sigma}\sqrt{d\log\left(1+\frac{nK}{d\sigma^2}\right)+2\log\left(\frac{1}{\delta}\right)} + \|\theta^*\|_2,$$

we have $P(\bar{\mathcal{E}}) \leq \delta$.

Proof. We start by defining some useful notations. For any $t = 1, 2, \ldots$, any $k = 1, 2, \ldots, \mathbf{K}_t$, we define

$$\eta_{t,k} = \mathbf{w}_t(\mathbf{a}_k^t) - \bar{w}(\mathbf{a}_k^t).$$

One key observation is that $\eta_{t,k}$'s form a Martingale difference sequence (MDS).[6] Moreover, since $\eta_{t,k}$'s are bounded in $[-1, 1]$ and hence they are conditionally sub-Gaussian with constant $R = 1$. We further define that

$$\begin{aligned}
\mathbf{V}_t =& \sigma^2 M_t = \sigma^2 I + \sum_{\tau=1}^{t}\sum_{k=1}^{\mathbf{K}_\tau} x_{\mathbf{a}_k^\tau} x_{\mathbf{a}_k^\tau}^T \\
\mathbf{S}_t =& \sum_{\tau=1}^{t}\sum_{k=1}^{\mathbf{K}_\tau} x_{\mathbf{a}_k^\tau}\eta_{t,k} = B_t - \sum_{\tau=1}^{t}\sum_{k=1}^{\mathbf{K}_\tau} x_{\mathbf{a}_k^\tau}\bar{w}(\mathbf{a}_k^t) = B_t - \left[\sum_{\tau=1}^{t}\sum_{k=1}^{\mathbf{K}_\tau} x_{\mathbf{a}_k^\tau} x_{\mathbf{a}_k^\tau}^T\right]\theta^*
\end{aligned}$$

As we will see later, we define $\mathbf{V}_t$ and $\mathbf{S}_t$ to use the "self normalized bound" developed in [1] (see Algorithm 1 of [1]). Notice that

$$M_t\bar{\theta}_t = \frac{1}{\sigma^2}B_t = \frac{1}{\sigma^2}\mathbf{S}_t + \frac{1}{\sigma^2}\left[\sum_{\tau=1}^{t}\sum_{k=1}^{\mathbf{K}_\tau} x_{\mathbf{a}_k^\tau} x_{\mathbf{a}_k^\tau}^T\right]\theta^* = \frac{1}{\sigma^2}\mathbf{S}_t + [M_t - I]\,\theta^*,$$

where the last equality is based on the definition of M_t. Hence we have

$$\bar{\theta}_t - \theta^* = M_t^{-1}\left[\frac{1}{\sigma^2}\mathbf{S}_t - \theta^*\right].$$

Thus, for any $e \in E$, we have

$$\begin{aligned}
\left|\langle x_e, \bar{\theta}_t - \theta^*\rangle\right| = \left|x_e^T M_t^{-1}\left[\frac{1}{\sigma^2}\mathbf{S}_t - \theta^*\right]\right| \leq \|x_e\|_{M_t^{-1}}\|\frac{1}{\sigma^2}\mathbf{S}_t - \theta^*\|_{M_t^{-1}} \\
\leq \|x_e\|_{M_t^{-1}}\left[\|\frac{1}{\sigma^2}\mathbf{S}_t\|_{M_t^{-1}} + \|\theta^*\|_{M_t^{-1}}\right],
\end{aligned}$$

where the first inequality follows from the Cauchy-Schwarz inequality and the second inequality follows from the triangle inequality. Notice that $\|\theta^*\|_{M_t^{-1}} \leq \|\theta^*\|_{M_0^{-1}} = \|\theta^*\|_2$, and $\|\frac{1}{\sigma^2}\mathbf{S}_t\|_{M_t^{-1}} = \frac{1}{\sigma}\|\mathbf{S}_t\|_{\mathbf{V}_t^{-1}}$ (since $M_t^{-1} = \sigma^2\mathbf{V}_t^{-1}$), so we have

$$\left|\langle x_e, \bar{\theta}_t - \theta^*\rangle\right| \leq \|x_e\|_{M_t^{-1}}\left[\frac{1}{\sigma}\|\mathbf{S}_t\|_{\mathbf{V}_t^{-1}} + \|\theta^*\|_2\right]. \tag{11}$$

Notice that the above inequality always holds. We now provide a high-probability bound on $\|\mathbf{S}_t\|_{\mathbf{V}_t^{-1}}$ based on "self normalized bound" proposed in [1]. From Theorem 1 of [1], we know that for any $\delta \in (0, 1)$, with probability at least $1 - \delta$, we have

$$\|\mathbf{S}_t\|_{\mathbf{V}_t^{-1}} \leq \sqrt{2\log\left(\frac{\det(\mathbf{V}_t)^{1/2}\det(\mathbf{V}_0)^{-1/2}}{\delta}\right)} \quad \forall t = 0, 1, \ldots$$

[6]Notice that the notion of "time" is indexed by the pair (t, k), and follows the lexicographical order.

Notice that $\det(\mathbf{V}_0) = \det(\sigma^2 I) = \sigma^{2d}$. Moreover, from the trace-determinant inequality, we have

$$[\det(\mathbf{V}_t)]^{1/d} \leq \frac{\text{trace}\,(\mathbf{V}_t)}{d} = \sigma^2 + \frac{1}{d}\sum_{\tau=1}^{t}\sum_{k=1}^{\mathbf{K}_\tau} \|x_{\mathbf{a}_k^t}\|_2^2 \leq \sigma^2 + \frac{tK}{d} \leq \sigma^2 + \frac{nK}{d},$$

where the second inequality follows from the assumption that $\|x_{\mathbf{a}_k^t}\|_2 \leq 1$ and $\mathbf{K}_\tau \leq K$, and the last inequality follows from $t \leq n$. Thus, with probability at least $1-\delta$, we have

$$\|\mathbf{S}_t\|_{\mathbf{V}_t^{-1}} \leq \sqrt{d\log\left(1+\frac{nK}{d\sigma^2}\right) + 2\log\left(\frac{1}{\delta}\right)} \quad \forall t = 0, 1, \ldots, n-1.$$

That is, with probability at least $1-\delta$, we have

$$\left|\langle x_e, \bar{\theta}_t - \theta^* \rangle\right| \leq \|x_e\|_{M_t^{-1}} \left[\frac{1}{\sigma}\sqrt{d\log\left(1+\frac{nK}{d\sigma^2}\right) + 2\log\left(\frac{1}{\delta}\right)} + \|\theta^*\|_2\right]$$

for all $t = 0, 1, \ldots, n-1$ and $\forall e \in E$. Recall that by definition of event $\mathcal{E}$, the above inequality implies that, if

$$c \geq \frac{1}{\sigma}\sqrt{d\log\left(1+\frac{nK}{d\sigma^2}\right) + 2\log\left(\frac{1}{\delta}\right)} + \|\theta^*\|_2,$$

then $P(\mathcal{E}) \geq 1-\delta$. That is, $P(\bar{\mathcal{E}}) \leq \delta$. ■

A.5 Conclude the Proof

Putting it together, for any $\sigma > 0$, any $\delta \in (0,1)$, and any

$$c \geq \frac{1}{\sigma}\sqrt{d\log\left(1+\frac{nK}{d\sigma^2}\right) + 2\log\left(\frac{1}{\delta}\right)} + \|\theta^*\|_2,$$

we have that

$$\begin{aligned} R(n) \leq & 2c\mathbb{E}\left[\sum_{t=1}^{n}\sum_{k=1}^{\mathbf{K}_t}\sqrt{x_{\mathbf{a}_k^t}^T M_{t-1}^{-1} x_{\mathbf{a}_k^t}}\middle|\mathcal{E}\right] + nKP(\bar{\mathcal{E}}) \\ \leq & 2cK\sqrt{\frac{dn\log\left[1+\frac{nK}{d\sigma^2}\right]}{\log\left(1+\frac{1}{\sigma^2}\right)}} + nK\delta. \end{aligned} \tag{12}$$

Choose $\delta = \frac{1}{nK}$, we have the following result: for any $\sigma > 0$ and any

$$c \geq \frac{1}{\sigma}\sqrt{d\log\left(1+\frac{nK}{d\sigma^2}\right) + 2\log(nK)} + \|\theta^*\|_2,$$

we have

$$R(n) \leq 2cK\sqrt{\frac{dn\log\left[1+\frac{nK}{d\sigma^2}\right]}{\log\left(1+\frac{1}{\sigma^2}\right)}} + 1.$$