APPENDIX

A TECHNICAL LEMMAS

A.1 FOLLOW-THE-REGULARIZED-LEADER TYPE RESULTS

Lemma 5. *Let $\{f_t\}_{t=1}^{\infty}$ be a sequence of functions and $\{x_t\}_{t=1}^{\infty} \subset \mathcal{K}$. Suppose there exists a sequence of lower barrier functions $\{h_t\}_{t=1}^{\infty}$ such that $h_t(x_t) = f_t(x_t)$ and $h_t \leq f_t$. Then, the following inequality holds:*

$$\max_{x\in\mathcal{K}} \sum_{t=1}^{T} f_t(x_t) - f_t(x) \leq \max_{x\in\mathcal{K}} \sum_{t=1}^{T} h_t(x_t) - h_t(x).$$

Proof. The proof follows from the inequalities:

$$\begin{aligned}\sum_{t=1}^{T} f_t(x_t) - f_t(x) \quad &= \sum_{t=1}^{T} h_t(x_t) - f_t(x) \\ &\leq \sum_{t=1}^{T} h_t(x_t) - h_t(x),\end{aligned}$$

and taking the maximum over $\mathcal{K}$. □

Lemma 6. *Let $\{f_t\}_{t=1}^{\infty}$ be a sequence of convex functions defined on a closed convex set $\mathcal{K}$, and let $\{x_t\}_{t=1}^{\infty}$ be a sequence of points in $\mathcal{K}$ such that the subgradient of f_t at x_t is denoted as g_t. Let $\{r_t\}_{t=1}^{\infty}$ be a sequence of non-negative convex functions. Then the update $x_{t+1} = \mathrm{argmin}_x\, g_{1:t}^T x + r_{0:t}(x)$ incurs regret at most*

$$\sum_{t=1}^{T} f_t(x_t) - f_t(x) \leq r_{0:T}(x) + \sum_{t=1}^{T} g_t^T(x_t - x_{t+1}).$$

Proof. The regret with respect to a fixed point x can be decomposed as follows:

$$\begin{aligned}&\sum_{t=1}^{T} f_t(x_t) - f_t(x) \\ &\quad\leq \sum_{t=1}^{T} g_t^T(x_t - x) \\ &\quad= \sum_{t=1}^{T} g_t^T(x_t - x_{t+1}) + g_t^T(x_{t+1} - x).\end{aligned}$$

The proof then follows from the inequality

$$\sum_{t=1}^{T} g_t^T x_{t+1} \leq r_{0:T}(x) + \sum_{t=1}^{T} g_t^T x,$$

which can be shown in a straightforward manner by induction. □

A.2 SMOOTHING AND UNBIASED GRADIENT ESTIMATES

Lemma 1. *Let $f\colon \mathbb{R}^n \to \mathbb{R}$, $A \in \mathbb{R}^{n\times n}$ be an SPSD matrix, and define $\widehat{f}(x) = \mathbb{E}_{v\sim\mathcal{B}^n}[f(x+Av)]$. Then, for $g_t = nf(x+Au)A^{-1}u$, the following holds: $\mathbb{E}_{u\sim\mathcal{S}^n}[g_t] = \nabla\widehat{f}(x)$.*

Proof.

$$\begin{aligned}\mathbb{E}_{u\sim\mathcal{S}^n}[g_t] &= \mathbb{E}_{u\sim\mathcal{S}^n}\left[nf(x+Au)A^{-1}u\right] \\ &= A^{-1}\mathbb{E}_{u\sim\mathcal{S}^n}\left[nf(x+Au)u\right] \\ &= A^{-1}\mathbb{E}_{v\sim\mathcal{B}^n}\left[\nabla_x f(x+Av)A\right] \\ &\quad \text{(by the divergence theorem)} \\ &= \nabla_x \mathbb{E}_{v\sim\mathcal{B}^n}\left[f(x+Av)\right]\end{aligned}$$

□

Lemma 4. *Let A be an SPSD matrix, and let $f\colon\mathbb{R}^n\to\mathbb{R}$ be A-strongly convex. Then $\widehat{f}$ is also A-strongly convex.*

Proof.

$$\begin{aligned}&\widehat{f}(x) - \widehat{f}(y) \\ &\quad= \mathbb{E}_{v\sim\mathcal{B}^n}\left[f(x+Av) - f(y+Av)\right] \\ &\quad\geq \mathbb{E}_{v\sim\mathcal{B}^n}\left[\nabla f(y+Av)^T(x-y) + \frac{1}{2}\|x-y\|_A^2\right] \\ &\quad= \nabla\mathbb{E}_{v\sim\mathcal{B}^n}\left[f(y+Av)\right]^T(x-y) + \frac{1}{2}\|x-y\|_A^2 \\ &\quad= \nabla\widehat{f}(y)^T(x-y) + \frac{1}{2}\|x-y\|_A^2\end{aligned}$$

□

A.3 AN INEQUALITY CONCERNING NORMALIZED SUMS

Lemma 7. *Let $\alpha_t \geq 0$, $\gamma > 0$, $\beta > 1$, and $\eta_t = \beta^{\frac{1}{1+\gamma}}(\alpha_{1:t})^{\frac{-1}{1+\gamma}}$. Then*

$$\left(\sum_{t=1}^{T} \eta_t^{\gamma}\alpha_t\right) + \frac{\beta}{\eta_T} \leq (2+\gamma)\beta^{\frac{\gamma}{1+\gamma}}\left(\alpha_{1:T}\right)^{\frac{1}{1+\gamma}}$$

Proof. By our choice of η_t, it follows that $\frac{\beta}{\eta_T} \leq \beta^{\frac{\gamma}{1+\gamma}}(\alpha_{1:T})^{\frac{1}{1+\gamma}}$. We now proceed by induction for the remaining expression. For $T = 1$, the inequality holds by direct inspection. If the statement is true for $T-1$, then

$$\begin{aligned}\sum_{t=1}^{T} \eta_t^{\gamma} \alpha_t &= \left(\sum_{t=1}^{T-1} \eta_t^{\gamma} \alpha_t\right) + \eta_T^{\gamma} \alpha_T \\ &\leq (1+\gamma)\beta^{\frac{\gamma}{1+\gamma}} \left(\alpha_{1:T-1}\right)^{\frac{1}{1+\gamma}} + \eta_T^{\gamma} \alpha_T \\ &= (1+\gamma)\beta^{\frac{\gamma}{1+\gamma}} \left(\alpha_{1:T} - \alpha_T\right)^{\frac{1}{1+\gamma}} + \frac{\beta^{\frac{\gamma}{1+\gamma}} \alpha_T}{\alpha_{1:T}^{\frac{\gamma}{1+\gamma}}} \\ &\leq (1+\gamma)\beta^{\frac{\gamma}{1+\gamma}} \alpha_{1:T}^{\frac{1}{1+\gamma}}\end{aligned}$$

since the second to last expression is optimized for $\alpha_T = 0$. □

A.4 FACTS ABOUT RANDOM SAMPLING

Lemma 8. *Let $x \sim \mathcal{D}$ be a random vector and A be a symmetric matrix. Then, the following identity holds:*

$$\mathbb{E}_{x\sim\mathcal{D}}[x^T A x] = \mathit{trace}(A\mathit{cov}(x)) + \mathbb{E}[x]^T A \mathbb{E}[x],$$

where $cov(x) = \mathbb{E}[xx^T] - \mathbb{E}[x]\mathbb{E}[x]^T$ is the covariance matrix associated to x.

Proof. The identity follows from

$$\begin{aligned}\mathbb{E}_{x\sim\mathcal{D}}[x^T A x] &= \mathbb{E}_{x\sim\mathcal{D}}[\text{trace}(Axx^T)] \\ &= \text{trace}\left(A\mathbb{E}_{x\sim\mathcal{D}}[xx^T]\right) \\ &= \text{trace}\left(A\left(\text{cov}(x) + \mathbb{E}[x]\mathbb{E}[x]^T\right)\right) \\ &= A\text{cov}(x) + \mathbb{E}[x]^T A \mathbb{E}[x],\end{aligned}$$

using the linearity of expectation and that of the trace operator. □

Lemma 9. *Let $u \sim \mathcal{S}^n$. Then $cov(u) = \frac{1}{n} I$ and $\mathbb{E}[u] = 0$.*

Proof. By symmetry, $(u_1, \ldots, u_i, \ldots, u_n)$ and $(u_1, \ldots, -u_i, \ldots, u_n)$ admit the same distribution. This implies that for all i, $\mathbb{E}[u_i] = \mathbb{E}[-u_i] = 0$ and also that the two random vectors admit the same covariance matrix. The latter means that $\mathbb{E}[u_i u_j] = \mathbb{E}[-u_i u_j] = 0$ for $i \neq j$.

Finally, the fact that u is distributed over the unit sphere implies that $\mathbb{E}[\sum_{i=1}^n u_i^2] = \sum_{i=1}^n \mathbb{E}[u_i^2] = 1$. By spherical symmetry, the elements of the vector are exchangable, so that $\mathbb{E}[u_i^2] = \mathbb{E}[u_j^2]$ for all $i, j \in \{1, \ldots, n\}$, which shows that $\mathbb{E}[u_i^2] = \frac{1}{n}$. □

B AdaBCO-Lipschitz REGRET BOUND

We present here the proof of Theorem 3, the regret bound for Algorithm 3.

Theorem 3 (AdaBCO using dynamic Lipschitz bounds)**.** *Let $\mathcal{K}$ be a convex set and $\mathcal{R}$ a ν-self-concordant barrier over $\mathcal{K}$. Assume that $|f| \leq C$. Then Algorithm 2 provides the regret bound:*

$$\begin{aligned}&\sum_{t=1}^{T} \mathbb{E}[f_t(y_t) - f_t(x)] \\ &\leq \mathbb{E}\left[5(\nu \log(T))^{\frac{1}{4}} \left(\sum_{t=1}^{T} \left(L_t n C^2 \sum_{j=1}^{n} \lambda_j(B_t)\right)^{\frac{1}{3}}\right)^{\frac{3}{4}}\right]\end{aligned}$$

Proof. We will first prove the intermediate inequality:

$$\begin{aligned}&\sum_{t=1}^{T} \mathbb{E}[f_t(y_t) - f_t(x)] \\ &\leq \left(\sum_{t=1}^{T} \mathbb{E}\left[L_t \delta_t \Big(\frac{1}{n}\sum_{j=1}^{n} \lambda_j(B_t)\Big)^{\frac{1}{2}}\right]\right) \\ &\quad + \mathbb{E}\left[\left(\sum_{t=1}^{T} \frac{\eta_t}{\delta_t^2}(n f_t(x_t + B_t u))^2\right) + \frac{1}{\eta_T}\nu \log(T)\right]\end{aligned}$$

As in the smooth scenario, we can compute that

$$\begin{aligned}&\mathbb{E}[\text{Reg}_T(w)] \\ &= \sum_{t=1}^{T} \mathbb{E}[f_t(y_t) - f_t(w)] \\ &= \sum_{t=1}^{T} \mathbb{E}[f_t(y_t) - f_t(x_t)] + \mathbb{E}[f_t(x_t) - \widehat{f}_t(x_t)] \\ &\quad + \mathbb{E}[\widehat{f}_t(w) - f_t(w)] + \mathbb{E}[\widehat{f}_t(x_t) - \widehat{f}_t(w)].\end{aligned}$$

By appealing to Theorem 1, it suffices to bound the first three terms using the L_t-Lipschitz property.

For the first term, we can write

$$\begin{aligned}&\mathbb{E}[f_t(y_t) - f_t(x_t)] \\ &= \mathbb{E}[\mathbb{E}_{u\sim\mathcal{S}^n}[f_t(x_t + \delta_t B_t u) - f_t(x_t)|x_t]] \\ &\leq \mathbb{E}[\mathbb{E}_{u\sim\mathcal{S}^n}[L_t \delta_t \|B_t u\|_2 | x_t]] \\ &\quad \text{(by } L_t\text{-Lipschitz)} \\ &\leq \mathbb{E}[L_t \delta_t \sqrt{\mathbb{E}_{u\sim\mathcal{S}^n}[u^T B_t^2 u | x_t]}] \\ &= \mathbb{E}\left[L_t \delta_t \Big(\frac{1}{n}\sum_{j=1}^{n} \lambda_j(B_t^2)\Big)^{\frac{1}{2}}\right] \\ &\quad \text{(by Lemmas 8 and 9).}\end{aligned}$$

The second term can be bounded using Jensen's inequality:

$$\begin{aligned}&\mathbb{E}[f_t(x_t) - \widehat{f}_t(x_t)] \\ &= \mathbb{E}\left[f_t(x_t) - \mathbb{E}_{v\sim\mathcal{B}^n}\left[f_t(x_t + Av)\right]\right] \\ &\leq \mathbb{E}\left[f_t(x_t) - f_t\left(\mathbb{E}_{v\sim\mathcal{B}^n}\left[x_t + Av\right]\right)\right] \\ &= 0.\end{aligned}$$

The third term can be bounded in a way similar to the first term:

$$\begin{aligned}
&\mathbb{E}[\widehat{f}_t(w) - f_t(w)] \\
&\quad = \mathbb{E}[\mathbb{E}_{v\sim\mathcal{B}^n}[f_t(w + \delta_t B_t v)] - f_t(w)] \\
&\quad \leq \mathbb{E}\left[\mathbb{E}_{v\sim\mathcal{B}^n}\left[L_t \|\delta_t B_t v\|_2\right]\right] \\
&\quad \leq \mathbb{E}\left[L_t \delta_t \Big(\frac{1}{n}\sum_{j=1}^{n} \lambda_j(B_t^2)\Big)^{\frac{1}{2}}\right].
\end{aligned}$$

Combining the estimates yields the intermediate inequality:

$$\begin{aligned}
&\sum_{t=1}^{T} \mathbb{E}[f_t(y_t) - f_t(x)] \\
&\quad \leq \left(\sum_{t=1}^{T} \mathbb{E}\left[L_t \delta_t \Big(\frac{1}{n}\sum_{j=1}^{n} \lambda_j(B_t)\Big)^{\frac{1}{2}}\right]\right) \\
&\qquad + \mathbb{E}\left[\left(\sum_{t=1}^{T} \frac{\eta_t}{\delta_t^2}(n f_t(x_t + B_t u))^2\right) + \frac{1}{\eta_T}\nu \log(T)\right] \\
&\quad \leq \left(\sum_{t=1}^{T} \mathbb{E}\left[L_t \delta_t \Big(\frac{1}{n}\sum_{j=1}^{n} \lambda_j(B_t)\Big)^{\frac{1}{2}}\right]\right) \\
&\qquad + \mathbb{E}\left[\left(\sum_{t=1}^{T} \frac{\eta_t}{\delta_t^2} n^2 C^2\right) + \frac{1}{\eta_T}\nu \log(T)\right]
\end{aligned}$$

and by our choice of δ_t, it follows that

$$\begin{aligned}
&\sum_{t=1}^{T} \mathbb{E}[f_t(y_t) - f_t(x)] \\
&\quad \leq \mathbb{E}\left[\sum_{t=1}^{T} 2\left(2\eta_t n C^2 L_t \sum_{j=1}^{n} \lambda_j(B_t)\right)^{1/3}\right] \\
&\qquad + \mathbb{E}\left[\frac{1}{\eta_T}\nu \log(T)\right].
\end{aligned}$$

Finally, our choice of η_t, Lemma 7, and the fact that $\eta_t \leq \eta_{t-1}$ yield:

$$\begin{aligned}
&\sum_{t=1}^{T} \mathbb{E}[f_t(y_t) - f_t(x)] \\
&\leq \mathbb{E}\left[5(\nu \log(T))^{\frac{1}{4}} \left(\sum_{t=1}^{T} \left(L_t n C^2 \sum_{j=1}^{n} \lambda_j(B_t)\right)^{\frac{1}{3}}\right)^{\frac{3}{4}}\right]
\end{aligned}$$

□