8 SUPPLEMENTARY

8.1 INFLUENCE MAXIMIZATION UNDER PROGRESSIVE MODEL: A BRIEF REVIEW

CELF method of Leskovec et al. [8] attempts to speed up the original greedy method, proposed by Kempe et al. [2], by reducing the number of calls to Monte Carlo routine for spread computation. CELF lazy method is based on the submodularity of the influence spread and can be applied to any submodular maximization problem. Although lazy evaluation improves the running time of the original greedy method by up to 700 times [8], it still does not scale to large graphs [4].

Recently heuristics have been proposed to approximate influence spread for LT [4] and IC [5] which enables the greedy method to scale for large networks. Chen et al. [4] suggest to use a local directed acyclic graph (LDAG) per node, instead of considering the whole graph, to approximate the influence flowing to the node. Goyal et al. propose SIMPATH method [9] under the LT model which is built on CELF method [8]. They approximate the influence spread by enumerating the simple paths starting from the seeds within a small neighborhood. Both of these methods have parameters to be tuned which control the trade-off between running time and accuracy of influence spread estimation. Methods presented in [4, 9] accelerate the greedy method [2] substantially and achieve high performance in influence maximization.

Gomez-Rodriguez et al. [6] propose a progressive continuous time influence model with dynamics similar to IC and show that influence maximization is NP-hard for this model as well. They show submodularity of influence spread and exploit the same greedy algorithm. In contrast to all other progressive models, influence spread has a closed form for this model but the computation is not scalable for large scale networks. A recent work [7] has scaled influence computation by developing a randomized algorithm for approximating it.

8.2 PROOF OF THEOREM 1

For proving this theorem we need the following lemmas.

Lemma 1. *When an interior node s is added to the current absorbing set $\mathcal{S}$, the new fundamental matrix F can be calculated from the previous one using the following equation:*

$$F_{ij}^{\mathcal{S}\cup\{s\}} = F_{ij}^{\mathcal{S}} - \frac{F_{is}^{\mathcal{S}} F_{sj}^{\mathcal{S}}}{F_{ss}^{\mathcal{S}}},$$

Proof. The proof is straightforward based on Schur complement theorem [30]. $\square$

This lemma helps avoiding the matrix inversion required for computing the new $F^{\mathcal{S}\cup\{s\}}$ whenever an interior node s is added to the seed set $\mathcal{S}$.

Lemma 2. *The expected number of passages through an interior node and the expected number of passages through its interior neighbors has the following relation:*

$$F_{ij}^{\mathcal{S}} = \begin{cases} \sum_k F_{ik}^{\mathcal{S}} R_{kj}^{\mathcal{S}} & i \neq j \\ 1 + \sum_k F_{ik}^{\mathcal{S}} R_{kj}^{\mathcal{S}} & i = j \end{cases}$$

Proof. We know $F^S = (I - R^S)^{-1}$. Start with $(I - R^S)^{-1}(I - R^S) = I$ and after multiplication and rearranging we get to the lemma's statement: $F^S = I + F^S R^S$ $\square$

Lemma 3. *Starting from node i the absorption probability by node s, when $S \cup \{s\}$ is the absorbing set, can be obtained from the expected number of passages through node s when it was not absorbing:*

$$Q_{is}^{\mathcal{S}\cup\{s\}} = \frac{F_{is}^{\mathcal{S}}}{F_{ss}^{\mathcal{S}}}. \tag{14}$$

Proof.

$$\begin{aligned} Q_{is}^{\mathcal{S}\cup\{s\}} &= \sum_{j\in\mathcal{V}\setminus\{\mathcal{S}\cup\{s\}\}} F_{ij}^{\mathcal{S}\cup\{s\}} B_{js}^{\mathcal{S}\cup\{s\}} \\ &= \sum_{j\in\mathcal{V}\setminus\{\mathcal{S}\}} F_{ij}^{\mathcal{S}\cup\{s\}} R_{js}^{\mathcal{S}} \\ &= \sum_{j\in\mathcal{V}\setminus\{\mathcal{S}\}} (F_{ij}^{S} - \frac{F_{is}^{S} F_{sj}^{S}}{F_{ss}^{S}}) R_{js}^{S} \\ &= \sum_{j\in\mathcal{V}\setminus\{\mathcal{S}\}} F_{ij}^{S} R_{js}^{S} - \frac{F_{is}^{S}}{F_{ss}^{S}} \sum_{j\in\mathcal{V}\setminus\{\mathcal{S}\}} F_{sj}^{S} R_{js}^{S} \\ &= F_{is}^{S} - \frac{F_{is}^{S}}{F_{ss}^{S}}(F_{ss}^{S} - 1) \\ &= \frac{F_{is}^{S}}{F_{ss}^{S}}, \end{aligned}$$

where the third and fifth equalities come from lemma 1 and lemma 2 respectively. $\square$

Proof of Theorem 1 is simply an instantiation of Lemma 3 for the case that we add node s as the first seed to the network and get $Q_{is}^{\{s\}} = \frac{F_{is}^{\emptyset}}{F_{ss}^{\emptyset}}$, where $\emptyset$ emphasizes that the bias node is the only boundary. Note that all of the three lemmas are general in a sense that absorbing set can contain any type of boundary points, including zero-value node like the bias node and one-value node like a seed node.

8.3 PROOF OF THEOREM 2

Proof. Consider an instance of the NP-complete Vertex Cover problem defined by an undirected and unweighted n-node graph $G = (\mathcal{V}, \mathcal{E})$ and an integer k; we want to know if there is a set $\mathcal{S}$ of k nodes in G so that every edge has at least one endpoint in $\mathcal{S}$. We show that this can be viewed as a special case of the influence maximization (9). Given an instance of the Vertex Cover problem involving a graph G, we define a corresponding instance of the influence maximization problem under HC for *infinite time horizon*, by considering the following settings in (1): (i) $\omega_{ij} = \omega_{ji} = 1$, if edge $(i - j) \in \mathcal{E}$, otherwise $\omega_{ij} = \omega_{ji} = 0$, (ii) bias node's value is zero $b = 0$, and (iii) β_i for all i's are equal to a known β. Note that since each interior node is connected to the zero-value bias node with edge weight β it cannot have value larger than $1 - \beta$. Hence, if there is a vertex cover $\mathcal{S}$ of size k in G, then one can deterministically make $\sigma(\mathcal{A}, \infty) = k + (n-k)(1-\beta)$ by targeting the nodes in the set $\mathcal{A} = \mathcal{S}$; conversely, this is the only way to get a set $\mathcal{A}$ with $\sigma(\mathcal{A}, \infty) = k + (n-k)(1-\beta)$. $\square$

8.4 PROOF OF THEOREM 3

As mentioned in Section 4.3 when $t \to \infty$ superposition principle applies for the HC model. We exploit this fact to prove the submodularity of influence spread. First note that $\sigma(\mathcal{S}, \infty)$ computed from (8) is the sum of node values and since the conic combination of submodular functions is also submodular it is enough to show that each node value, i.e., $v(i)$ is submodular to proof Theorem 3. Here we need to work with the general set of bias nodes (compare to single bias node b) which we call ground set $\mathcal{G}$. We introduce a new notation where the value of node i is shown with $v^{\mathcal{S},\mathcal{G}}(i)$. Also seed nodes can have arbitrary value of $\geq b$ instead of all 1 values.For proving the submodularity of $v(i)$ we should prove:

$$v^{\mathcal{T}\cup\{s\},\mathcal{G}}(i) - v^{\mathcal{T},\mathcal{G}}(i) \geq v^{\mathcal{S}\cup\{s\},\mathcal{G}}(i) - v^{\mathcal{S},\mathcal{G}}(i), \mathcal{T} \subseteq \mathcal{S} \tag{15}$$

We invoke superposition to perform the subtraction:

$$v^{\{s_{v_L}\},\mathcal{G}\cup\mathcal{T}}(i) \geq v^{\{s_{v_R}\},\mathcal{G}\cup\mathcal{S}}(i), \qquad \mathcal{T} \subseteq \mathcal{S} \tag{16}$$

where v_L and v_R emphasize that the value of the new seed node is different in left and right hand side and is qual to $v_L = \left(1 - v^{\mathcal{T},\mathcal{G}}(s)\right)$ and $v_R = \left(1 - v^{\mathcal{S},\mathcal{G}}(s)\right)$. Note that $v_L \geq v_R$ since $\mathcal{T} \subseteq \mathcal{S}$. We can not compare the value of nodes in two different networks unless they share same grounds and seeds with possibly different values for each seed. Therefore, we try to make the grounds of both sides of (16) identical by expanding the LHS of (16) using superposition law [31]:

$$v^{\{s_{v_L}\},\mathcal{G}\cup\mathcal{T}}(i) = v^{\{s_{v_L}\},\mathcal{G}\cup\mathcal{S}}(i) + v^{\mathcal{D},\mathcal{G}\cup\mathcal{S}\cup s,}(i) \tag{17}$$

where $\mathcal{D} = \mathcal{S} - \mathcal{T}$. Although second term of (17) is complicated but for our analysis it is enough to note that it is a non-negative number $\alpha \geq 0$. Now the submodularity inequality (15) reduces to:

$$v^{\{s_{v_L}\},\mathcal{G}\cup\mathcal{S}}(i) + \alpha \geq v^{\{s_{v_R}\},\mathcal{G}\cup\mathcal{T}}(i) \tag{18}$$

Now both sides have the same set of sources and grounds and we now $v_L(u) \geq v_R(u)$ and $\alpha \geq 0$ which completes the proof.