A PROOFS

A.1 PROOF OF PROPOSITION 3.4

Proof. (Proof of Proposition 3.4)

Consider the most basic case when there is only one Boolean variable \( b \) in theory \( \theta \). Let \( \theta' \) be an SMT(\( \mathcal{LRA} \)) theory defined as follow

\[
\theta' = \theta\{b : \lambda_b\} \land (-1 \leq \lambda_b \leq 0)
\]

where \( \theta\{b : \lambda_b\} \) is obtained by replacing all atom \( b \) by \( 0 < \lambda_b \) and replacing all its negation \( \neg b \) by \( \lambda_b < 0 \) in theory \( \theta \).

Recall that weight functions are defined by a set of literals \( \mathcal{L} \) and a set of per-literal weight functions \( \mathcal{P} = \{p_\ell(x)\}_{\ell \in \mathcal{L}} \). When a literal \( \ell \) is satisfied in a world, denoted by \( x \land b \models \ell \), weights are defined as follows

\[
w(x, b) = \prod_{\ell \in \mathcal{L}, x \land b \models \ell} p_\ell(x).
\]

Let \( \mathcal{L}' \) be a set of literals obtained by replacing Boolean literal \( b \) by \( 0 < \lambda_b \) and replacing its negation \( \neg b \) by \( \lambda_b < 0 \) in theory \( \theta \) as we do for theory. For the set of per-literal weight functions \( \mathcal{P}' \), we define it for introduced real variable \( \lambda_b \) by \( p(\lambda_b = 0) = p_b \) and \( p(\lambda_b < 0) = p_{\neg b} \).

Then we have that for any \( x^* \),

\[
w'(x^*, \lambda_b) = \begin{cases} w(x^*, b), & 1 > \lambda_b > 0 \\ w(x^*, \neg b), & -1 < \lambda_b < 0 \end{cases}
\]

By definition of WMII, we write WMII(\( \theta, w \mid x, b \)) in its integration form as follows.

\[
\text{WMII}(\theta, w \mid x, b) = \int_{\theta(x, b)} w(x, b) dx + \int_{\theta(x, \neg b)} w(x, \neg b) dx
\]

For the first term in the above equation, we can rewrite it s.t. Boolean variable \( b \) is replaced by real variable \( \lambda_b \) in the following way.

\[
\int_{\theta(x, b)} w(x, b) dx = \int_{0}^{1} \int_{\theta(x, b)} w(x, b) dxd\lambda_b = \int_{\theta'(x, \lambda_b)} w'(x, \lambda_b) dxd\lambda_b
\]

By doing this to the other integration term of WMII(\( \theta, w \mid x, b \)), and also by the definition of WMII, we finally obtain that

\[
\text{WMII}(\theta, w \mid x, b) = \text{WMII}(\theta', w' \mid x')
\]

where \( x' = x \cup \{\lambda_b\} \) is a set of real variables. The proof above can be easily adapted to multiple Boolean variable cases, which proves our proposition.

A.2 PROOF OF PROPOSITION 3.5

Proof. (Proof of Proposition 3.5) To start with, we consider SMT(\( \mathcal{LRA} \)) theory \( \theta \) with no Boolean variables with a simple weight function \( w \) where the set of literal \( \mathcal{L} = \{\ell\} \) has only one literal and literal weight function

\[
p_\ell(x) = \prod_{i=0}^{n} x_i^{p_i}.
\]

Let \( \theta' = (x \land (\ell \Rightarrow \theta_p)) \land (-\ell \Rightarrow \hat{\theta}_p) \) where \( p = p_\ell \) for brevity, \( \theta_p \) is as defined in Claim A.1 and \( \hat{\theta}_p := \bigwedge_{i=0}^{p_{\ell}} z \leq 1 \). Then we can rewrite WMII(\( \theta, w \mid x \)) as MI problem by Claim A.1 as follows.

\[
\text{WMII}(\theta, w \mid x) = \int_{\theta(x)} w(x) dx = \int_{\theta(x)} \text{MI}(\theta_p \mid z \mid x, z) + \int_{\theta(x)} \text{MI}(\hat{\theta}_p \mid \neg z \mid x, z)
\]

Take \( x' = x \cup z \) then the proposition holds. The proof can be easily adapted for monomials with non-trivial coefficient by inducing more real variables \( z \). It also holds for more general weight functions with literal set \( \mathcal{L} = \{\ell_i\}_{i=1}^{k} \) and set of monomial per-literal weight functions \( \mathcal{P} = \{p_{\ell_i}\}_{i=1}^{k} \), by taking theory \( \theta' \) as follows which completes the proof of proposition.

\[
\theta' = \theta \land \bigwedge_{i=1}^{k} (\ell_i \Rightarrow \theta_{p_{\ell_i}}) \land \bigwedge_{i=1}^{k} (\neg \ell_i \Rightarrow \hat{\theta}_{p_{\ell_i}}).
\]
Proof. (Proof of Claim A.1) By definition of theory \( \theta_f \),

\[
MI(\theta_f \mid z; x) = \int_{\theta_f(z)} 1dz = \prod_{i=1}^{n} p_i \prod_{j=1}^{n} \int_{x_i}^{x_i} 1dz_j
\]

\[
= \prod_{i=1}^{n} p_i \prod_{j=1}^{n} x_i = \prod_{i=1}^{n} x_i^{p_i} = f(x).
\]

\[\square\]

A.3 REDUCTION TO MI WITH POLYNOMIAL WEIGHTS

The reduction from WMI problems to MI problems in Proposition 3.5 can also be done for arbitrary polynomial weight functions but can increase treewidth of primal graphs. We give a formal description on this reduction as follows.

Let \( \theta \) be an SMT(\text{LRA}) theory with no Boolean variables with weight functions where the set of literal \( \ell = \{ \ell \} \) has only one literal and literal weight function is a polynomial, denoted by \( p(x) = \sum_{i=1}^{k} \alpha_i f_i(x) \) with each \( f_i \) a monomial function.

It has been shown in the proof of Proposition 3.5 in Section A.2 that for each monomial function \( f_i \), there exist two SMT(\text{LRA}) theories \( \theta_i \) and \( \hat{\theta}_i \) such that \( MI(\theta_i \mid z_i; x) = f_i(x) \) and \( MI(\hat{\theta}_i \mid z_i; x) = 1 \).

Let’s define theories \( \theta'_i = \theta_i \land (0 < v_i < \alpha_i) \) and \( \hat{\theta}'_i = \hat{\theta}_i \land (0 < v_i < 1) \) with parameter variables \( v_i \). Also define an indicator variable \( \lambda \) with real domain \([0, 1]\) and literals \( \ell_i = i - 1 < \lambda < i \) with \( i \in \{1, 2, \ldots, k\} \).

Then we have that for an SMT(\text{LRA}) theory \( \theta' \) defined as follows, it holds that \( WMI(\theta, w \mid x) = MI(\theta' \mid x, z) \) with \( z \) denoting all auxiliary variables.

\[\theta' = \theta \land (\ell \iff \bigvee_{i=1}^{k} (\ell_i \implies \theta'_i) \bigwedge_{i=1}^{k} (\neg \ell_i \implies \hat{\theta}'_i))\]

Why the WMI problem and the MI problem are equal can be proved by the following observations.

\[
WMI(\theta, w \mid x) = \int_{\theta(x)} w(x)dx
\]

\[
= \int_{\theta(x) \land \ell(x)} p(x)dx + \int_{\theta(x) \land \neg \ell(x)} 1dx
\]

For the first term in Equation 5, we have that

\[
\int_{\theta(x) \land \ell(x)} p(x)dx = \sum_{i=1}^{k} \int_{\theta(x) \land \ell(x)} \alpha_i f_i(x)dx = \sum_{i=1}^{k} \int_{\theta(x) \land \ell(x) \land \ell_i} \alpha_i f_i(x)d\lambda
\]

\[
= \sum_{i=1}^{k} \int_{\theta(x) \land \ell(x) \land \ell_i} 1dx dz = MI(\theta' \mid x, z)
\]

Also for the second term in Equation 5, it equates to \( MI(\theta' \land \neg \ell \mid x, z) \). Therefore, reduction from the WMI problem to the MI problem holds. Although the reduction process we show here is for theories with one polynomial weight function, this process can be generalized to theories with multiple polynomial weight functions with little modification.

A.4 PROOF OF PROPOSITION 4.1

Proof. (Proof of Proposition 4.1) It follows from definition of WMI. Denote the set of real variables \( x \setminus \{y\} \) by \( \hat{x} \). From the definition of WMI in Equation 2.2, we can obtain the following partial derivative of WMI of theory \( \theta \) w.r.t. variable \( y \).

\[\frac{\partial}{\partial y} WMI(\theta, w \mid x, b) \big|_{y=y^*} = \sum_{\mu \in B^m_{\theta(y^*, \hat{x}, \mu)}} \int_{\mu} w(y^*, \hat{x}, \mu) d\hat{x}\]

where the variable \( y \) is fixed to value \( y^* \) in weight function, \( \mu \) are total truth assignments to Boolean variables as defined before. The weight function is integrated over set \( \{\hat{x}^* \mid \theta(y^*, \hat{x}^*, \mu) \text{ is true}\} \). We define \( p(y) \) as follows

\[p(y) := \sum_{\mu \in B^m_{y, \theta(\hat{x}, \mu)}} \int_{\mu} w(y, \hat{x}, \mu) d\hat{x}\]

Since weight functions \( w \) are piecewise polynomial, function \( p(y) \) is a univariate piecewise polynomial \( p(y) \), and \( WMI(\theta, w \mid x, b) \) is an integration over \( p(y) \), which finishes our proof. \[\square\]

A.5 PROOF OF THEOREM 4.4

Claim A.2. For each path in the primal graph that starts with the root and ends with a leaf, and each real variable in path with height \( i \), its number of polynomial pieces is \( O(n \cdot c^{i+1}) \).
the search space is bounded by a variable has size n·c·h. Therefore, the size of the search space is bounded by O(l·(n^3 · c^h)^h_i).}

\[\text{Proof. (Proof of Theorem 4.4)}\]

Let \( p \) be an arbitrary path in the pseudo tree \( T \) that starts with the root and ends with a leaf. Denote the maximum polynomial degree in weight functions by \( d \). By Claim A.2 for each variable, it has at most \( O(n^3 · c^h) \) polynomial pieces. Moreover from Prop. 4.1, polynomials defined over each pieces have at most \( n(d + h) \) polynomial degree. Therefore the set of values chosen to do instantiation on a certain real variable has size \( O(n^3 · c^h) \) and each path \( p \) induces a search space with size \( O((n^3 · c^h)^{h_i}) \) since length of each path is bounded by \( h_i \).

The pseudo tree \( T \) is covered by \( l \) such directed paths. The union of their individual search spaces covers the whole search space, where every distinct full path in the search space appears exactly once. Therefore, the size of the search space is bounded by \( O(l · (n^3 · c^h)^{h_i}) \).