

Efficient Bayesian Inference for a Gaussian Process Density Model: Supplementary Material

A THE CONDITIONAL POSTERIOR POINT PROCESS

Here we prove that the conditional posterior point process in Equation (13) again is a Poisson process using Campbell's theorem [1, chap. 3]. For an arbitrary function $h(\cdot, \cdot)$ we set $H \doteq \sum_{(\mathbf{x}, \omega) \in \Pi} h(\mathbf{x}, \omega)$. We calculate the characteristic functional

$$\begin{aligned} \mathbb{E}_{\phi_\lambda} [e^H | g, \lambda] &= \\ &= \frac{\mathbb{E}_{\phi_\lambda} \left[\prod_{(\omega, \mathbf{x}) \in \Pi} e^{f(\omega, -g(\mathbf{x})) + h(\mathbf{x}, \omega)} \middle| g, \lambda \right]}{\exp \left(\int_{\mathcal{X} \times \mathbb{R}^+} (e^{f(\omega, -g(\mathbf{x}))} - 1) \phi_\lambda(\mathbf{x}, \omega) d\omega d\mathbf{x} \right)} = \\ &= \frac{\exp \left\{ \int_{\mathcal{X} \times \mathbb{R}^+} (e^{f(\omega, -g(\mathbf{x})) + h(\mathbf{x}, \omega)} - 1) \phi_\lambda(\mathbf{x}, \omega) d\omega d\mathbf{x} \right\}}{\exp \left(\int_{\mathcal{X} \times \mathbb{R}^+} (e^{f(\omega, -g(\mathbf{x}))} - 1) \phi_\lambda(\mathbf{x}, \omega) d\omega d\mathbf{x} \right)} = \\ &= \exp \left\{ \int_{\mathcal{X} \times \mathbb{R}^+} (e^{h(\mathbf{x}, \omega)} - 1) e^{f(\omega, -g)} \phi_\lambda(\mathbf{x}, \omega) d\omega d\mathbf{x} \right\} = \\ &= \exp \left\{ \int_{\mathcal{X} \times \mathbb{R}^+} (e^{h(\mathbf{x}, \omega)} - 1) \Lambda(\mathbf{x}, \omega) d\omega d\mathbf{x} \right\}, \end{aligned}$$

where the last equality follows from the definition of $\phi_\lambda(\mathbf{x}, \omega)$ and the tilted Pólya–Gamma density. Using the fact that a Poisson process is uniquely characterised by its generating function this shows that the conditional posterior $p(\Pi | g, \lambda)$ is a marked Poisson process.

B VARIATIONAL LOWER BOUND

The full variational lower bound is given by

$$\begin{aligned} \mathcal{L}(q) &= \sum_{n=1}^N \left\{ \mathbb{E}_Q [\ln \lambda] + \ln \pi(\mathbf{x}_n) + \mathbb{E}_Q [f(\omega_n, g(\mathbf{x}_n))] - \ln \cosh \left(\frac{c_n}{2} \right) + \frac{c_n^2}{2} \mathbb{E}_Q [\omega_n] \right\} \\ &+ \int_{\mathcal{X}} \int_{\mathbb{R}^+} \left\{ \mathbb{E}_Q [\ln \lambda] + \mathbb{E}_Q [f(\omega, -g(\mathbf{x}))] - \ln \lambda_1 - \ln \sigma(-c(\mathbf{x})) - \ln \cosh \left(\frac{c(\mathbf{x})}{2} \right) - \frac{c(\mathbf{x})^2}{2} \omega \right. \\ &\left. - \frac{c(\mathbf{x}) - g_1(\mathbf{x})}{2} + 1 \right\} \Lambda_1(\mathbf{x}, \omega) d\omega d\mathbf{x} - \mathbb{E}_Q [\lambda] + \mathbb{E}_Q \left[\ln \frac{p(\lambda)}{q(\lambda)} \right] + \mathbb{E}_Q \left[\ln \frac{p(\mathbf{g}_s)}{q(\mathbf{g}_s)} \right]. \end{aligned}$$

References

- [1] John Frank Charles Kingman. *Poisson processes*. Wiley Online Library, 1993.