Supplementary Material: Lifted Marginal MAP Inference

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1 Lemmas

We will start by proving Lemma 0, which will be used in the proof of Lemma 2 .

Lemma 0. *Given a function $f(x, y)$, let (x^*, y^*) be a maximizing assignment, i.e., $(x^*, y^*) = arg\max_{x,y} f(x, y)$. Then, $\forall y'$ s.t. $y^{'} = arg\max_y f(x^*, y)$, $(x^*, y^{'})$ is also a maximizing assignment.*

Proof. We can write the following (in)equality:

$$f(x^*, y^*) \leq \max_y f(x^*, y) = f(x^*, y')$$

But since $(x^*$,$y^*)$ was the maximizing assignment for $f(x, y)$, it must be the case that $f(x^*, y^*) = f(x^*, y')$. Hence, (x^*, y') must also be a maximizing assignment. Hence, proved.

Lemma 2. *Consider the MMAP problem over $M_{\tilde{X}}$. Let q_p be an assignment to the propositional MAP predicates. Let $M'_{\tilde{X}}$ be an MLN obtained by substituting the truth value in q_p for propositional predicates. Then, if $M'_{\tilde{X}}$ has a solution at extreme for all possible assignments of the form q_p then, $M_{\tilde{X}}$ also has a solution at extreme.*

Proof. The MMAP problem can be written as:

$$\arg\max_{q_p, q_u} \sum_{s_p, s_u} W_{M_{\tilde{X}}}(q_p, q_u, s_p, s_u) \tag{1}$$

here, q_p, q_u denote an assignment to the propositional and unary MAX predicate groundings in $M_{\tilde{X}}$, respectively. Similarly, s_p, s_u denote an assignment to the propositional and unary SUM predicate groundings in $M_{\tilde{X}}$, respectively. Let q_p^* denote an optimal assignment to the propositional MAX predicates. Then, using Lemma 0, we can get the MMAP assignment q_u^* as a solution to the following problem:

$$\arg\max_{q_u} \sum_{s_p, s_u} W_{M'_{\tilde{X}}}(q_u, s_u, s_p) \tag{2}$$

where $M'_{\tilde{X}}$ is obtained by substituting the truth assignment q_p^* in $M_{\tilde{X}}$. Since, $M'_{\tilde{X}}$ has a solution at extreme $\forall q_p$, it must also be at extreme when $q_p = q_p^*$. Hence, q_u^* must be at extreme. Hence, proved.

Lemma 5. *The solution to the MMAP formulation $arg\max_q W_{M_{\tilde{X}}}(q)$ lies at extreme iff solution to its equivalent formulation:*

$$arg\max_{N_1, N_2, \cdots, N_R} \sum_s \prod_{l=1}^{R} f_l(s)^{N_l} \tag{3}$$

subject to the constraints $\forall l, N_l \geq 0, N_l \in \mathbb{Z}$ and $\sum_l N_l = m$ lies at extreme.

Proof. If $\arg\max_{N_1,N_2,\cdots,N_R} \sum_s \prod_{l=1}^{R} f_l(s)^{N_l}$ lies at extreme then $\exists l$ such that $N_l = m$ and $N_{l'} = 0, \forall l' \neq l$. Let v_l denote the value taken by the groundings of the unary MAX predicates corresponding to index l. Since $N_l = m$, it must be the case that all the groundings get the identical value v_l. Hence, the solution to $\arg\max_q W_{M_{\tilde{X}}}(q)$ lies at extreme. Similar proof strategy holds for the other way around as well.

Lemma. ***(Induction Base Case in Proof of Lemma 6)**: Let $f_1(s), f_2(s)$ and $g(s)$ denote real-valued functions of a vector valued input s [1]. Further, let each of $f_1(s), f_2(s), g(s)$ be non-negative. Then, for $N \in \mathcal{R}$, we define a function $h(N) = \sum_s f_1(s)^N f_2(s)^{m-N} \times g(s)$ where the domain of h is further restricted to be in the interval $[0, m]$, i.e., $0 \leq N \leq m$. The maxima of the function $h(N)$ lies at $N = 0$ or $N = m$.*

Proof. First derivative of $h(N)$ with respect to N is:

$$\frac{dh}{dN} = \sum_s \Big(f_1(s)^N f_2(s)^{m-N} g(s) \times [log(f_1(s)) - log(f_2(s))] \Big)$$

[1]Recall that s was an assignment to all the propositional SUM predicates in our original Lemma.

Second derivative of $h(N)$ with respect to N is given as:

$$\frac{d^2h}{dN^2} = \sum_s \Big(f_1(s)^N f_2(s)^{m-N} g(s) \times [log(f_1(s)) - log(f_2(s))]^2 \Big) \geq 0$$

The inequality follows from the fact that each of f_1, f_2, g is non-negative. Hence, the second derivative of $h(N)$ is non-negative which means the function is convex. Therefore, the maximum value of this function must lie at the end points of its domain, i.e, either at $N = 0$ or at $N = m$.

Lemma 8. *Let M be an MLN and M^r be the reduced MLN with respect to the SOM-R equivalence class $\tilde{X}$. Let q and q^r denote two corresponding extreme assignments in M and M^r, respectively. Then, $\exists$ a monotonically increasing function g such that $W_M(q) = g(W_{M^r}(q^r))$.*

Proof. First, we note that if we multiply the weight w_i of a formula f_i in an MLN by a factor k, then, the corresponding potential ϕ_{ij} (i.e., potential corresponding to the j^{th} grounding of the i^{th} formula) gets raised to the power k. If w_i gets replaced by $k \times w_i$, then, correspondingly, ϕ_{ij} gets replaced by $(\phi_{ij})^k$ $\forall j$. We will use this fact in the following proof.

As in the case of Lemma 7, we will instead work with the variablized MLNs $M_{\tilde{X}}$ and $M^r_{\tilde{X}}$, respectively. Let $q = (q_p, q_u)$ be the MMAP assignment for $\mathcal{Q}$ in $M_{\tilde{X}}$ and similarly $q = (q^r_p, q^r_u)$ be the MMAP assignment for $\mathcal{Q}$ in $M^r_{\tilde{X}}$.

For MLN $M_{\tilde{X}}$, the MMAP objective W_M at (q_p, q_u) can be written as $W_{M_{\tilde{X}}}(q_p, q_u) =$:

$$\sum_{s_p, s_u} \Big(\prod_{i=1}^{r} \prod_{j=1}^{m} \phi_{ij}(q_p, q_u, s_p, s_u) \prod_{k=1}^{t} \phi_k(q_p, s_p) \Big) \quad (4)$$

where ϕ_{ij} are potentials over formulas containing some $X \in \tilde{X}$ and ϕ_k are potentials over formulas which do not contain any $X \in \tilde{X}$. In particular, note that we have separated out the formulas which involve a variable from the class $\tilde{X}$ from those which don't. r denotes the count of the formulas of the first type and t denotes the count of the formulas of the second type. We will use this form in the following proof.

Let the reduced domain of $\tilde{X}$ in M^r is given by $\{x_1\}$, i.e., the only constant which remains in the domain is corresponding to index $j = 1$. Next we prove the above lemma for the two cases considered in Definition 5:

CASE 1: $\forall P \in \mathcal{S}, P$ contains a variable from $\tilde{X}$
In this case $M_{\tilde{X}}$ and $M^r_{\tilde{X}}$ will not contain any propositional `SUM` predicate i.e. $s_p = \emptyset$.

In this case, while constructing $M^r_{\tilde{X}}$, for formulas not containing some $X \in \tilde{X}$ we divided the weight by m. This combined with the result stated in the beginning of this proof, the MMAP objective for $M^r_{\tilde{X}}$ can be written as:

$$\begin{aligned} W_{M^r_{\tilde{X}}}(q_p, q_u) &= \sum_{s_u} \Big(\prod_{i=1}^{r} \phi_{i1}(q_p, q_{u_1}, s_{u_1}) \prod_{k=1}^{t} \phi_k(q_p)^{\frac{1}{m}} \Big) \\ &= \Big(\sum_{s_{u_1}} \prod_{i=1}^{r} \phi_{i1}(q_p, q_{u_1}, s_{u_1}) \Big) \prod_{k=1}^{t} \phi_k(q_p)^{\frac{1}{m}} \end{aligned}$$

Next for MLN $M_{\tilde{X}}$ we have, $W_{M_{\tilde{X}}}(q_p, q_u) =$

$$\begin{aligned} &\sum_{s_u} \Big(\prod_{i=1}^{r} \prod_{j=1}^{m} \phi_{ij}(q_p, q_u, s_u) \prod_{k=1}^{t} \phi_k(q_p) \Big) \\ &= \Big(\sum_{s_u} \prod_{j=1}^{m} \prod_{i=1}^{r} \phi_{ij}(q_p, q_u, s_u) \Big) \prod_{k=1}^{t} \phi_k(q_p) \\ &= \Big(\sum_{s_{u_j}} \prod_{j=1}^{m} \prod_{i=1}^{r} \phi_{ij}(q_p, q_{u_j}, s_{u_j}) \Big) \prod_{k=1}^{t} \phi_k(q_p) \\ &= \Big(\prod_{j=1}^{m} \sum_{s_{u_j}} \prod_{i=1}^{r} \phi_{ij}(q_p, q_{u_j}, s_{u_j}) \Big) \prod_{k=1}^{t} \phi_k(q_p) \\ &= \Big(\prod_{j=1}^{m} \sum_{s_{u_j}} \prod_{i=1}^{r} \phi_{ij}(q_p, q_{u_1}, s_{u_j}) \Big) \prod_{k=1}^{t} \phi_k(q_p) \\ &= \Big(\sum_{s_{u_j}} \prod_{i=1}^{r} \phi_{ij}(q_p, q_{u_1}, s_{u_j}) \Big)^m \prod_{k=1}^{t} \phi_k(q_p) \\ &= \Big(\sum_{s_{u_j}} \prod_{i=1}^{r} \phi_{ij}(q_p, q_{u_1}, s_{u_j}) \Big)^m \prod_{k=1}^{t} \Big(\phi_k(q_p)^{\frac{1}{m}} \Big)^m \\ &= \Big(\sum_{s_{u_j}} \prod_{i=1}^{r} \phi_{ij}(q_p, q_{u_1}, s_{u_j}) \Big)^m \Big(\prod_{k=1}^{t} \phi_k(q_p)^{\frac{1}{m}} \Big)^m \\ &= \Big(\sum_{s_{u_j}} \prod_{i=1}^{r} \phi_{ij}(q_p, q_{u_1}, s_{u_j}) \prod_{k=1}^{t} \phi_k(q_p)^{\frac{1}{m}} \Big)^m \\ &= \Big(W_{M^r_{\tilde{X}}}(q_p, q_u) \Big)^m \end{aligned}$$

First equality comes by removing s_p from Equation 4. In second equality we switch the order of two products. In third equality we have made explicit the dependence of ϕ_{ij} on q_{u_j} and s_{u_j} i.e. groundings corresponding to j^{th} constant. In fourth equality we use inversion elimination (de Salvo Braz, Amir, and Roth 2005) to invert the sum over s_{u_j} and product over j. Next, since $\tilde{X}$ is SOM-R, from Theorem 1 we know q_u lies at extreme i.e. $\forall j$, $q_{u_j} = q_{u_1}$, so we replace all q_{u_j} by q_{u_1} in fifth equality. Next, after summing out s_{u_j} all ϕ_{ij} will behave identically [2], so we reduce $\prod_{j=1}^{m}$ to exponent m. In the next steps we do basic algebraic manipulations to show $W_{M_{\tilde{X}}}(q) = g(W_{M^r_{\tilde{X}}}(q^r))$ where g is function defined as $g(x) = x^m$, i.e., g is monotonically increasing. Hence, proved.

CASE 2: $\forall P \in \mathcal{S}$, P doesn't contain a variable from $\tilde{X}$ In this case $M_{\tilde{X}}$ will not contain any unary `SUM` predicate i.e. $s_u = \emptyset$.

In this case for the reduced MLN $M^r_{\tilde{X}}$ we multiply the weight of formulas containing some $X \in \tilde{X}$ by m and domain of $\tilde{X}$ is reduced to a single constant. Combining this fact along with the result shown in the beginning of this proof, MAP objective for $M^r_{\tilde{X}}$ is given by:

$$W_{M^r_{\tilde{X}}}(q_p, q_u) = \sum_{s_p} \Big(\prod_{i=1}^{r} \phi_{i1}(q_p, q_{u_1}, s_p)^m \prod_{k=1}^{t} \phi_k(q_p, s_p) \Big)$$

Next for MLN $M_{\tilde{X}}$ we have, $W_{M_{\tilde{X}}}(q_p, q_u) =$

$$\begin{aligned}
&\sum_{s_p} \Big(\prod_{i=1}^{r} \prod_{j=1}^{m} \phi_{ij}(q_p, q_u, s_p) \prod_{k=1}^{t} \phi_k(q_p, s_p) \Big) \\
&= \sum_{s_p} \Big(\prod_{i=1}^{r} \prod_{j=1}^{m} \phi_{ij}(q_p, q_{u_j}, s_p) \prod_{k=1}^{t} \phi_k(q_p, s_p) \Big) \\
&= \sum_{s_p} \Big(\prod_{i=1}^{r} \prod_{j=1}^{m} \phi_{ij}(q_p, q_{u_1}, s_p) \prod_{k=1}^{t} \phi_k(q_p, s_p) \Big) \\
&= \sum_{s_p} \Big(\prod_{i=1}^{r} (\phi_{i1}(q_p, q_{u_1}, s_p))^m \prod_{k=1}^{t} \phi_k(q_p, s_p) \Big) \\
&= W_{M^r_{\tilde{X}}}(q_p, q_u)
\end{aligned}$$

First equality comes by removing s_u from Equation 4. In second equality we have made explicit the dependence of ϕ_{ij} on q_{u_j} i.e. groundings corresponding to j^{th} constant. Next, since $\tilde{X}$ is SOM-R, from Theorem 1 we know q_u lies at extreme i.e. $\forall j$, $q_{u_j} = q_{u_1}$, so we replace all q_{u_j} by q_{u_1} in third equality. Last equality comes from the fact that ϕ_{ij}'s are identical to each other up to renaming of the index j as argued earlier. Hence we can write $\prod_j \phi_{ij}$ as $(\phi_{i1})^m$. Hence, in this case, we have $W_{M_{\tilde{X}}}(q) = W_{M^r_{\tilde{X}}}(q^r)$ implying that the function g is identity (and hence, monotonically increasing).

From proofs of Case 1 and Case 2 we conclude that $\exists$ a monotonically increasing function g such that $W_M(q) = g(W_{M^r}(q^r))$.

[2] ϕ_{ij}'s are identical to each other up to renaming of the index j, due to the normal form assumption.

References

de Salvo Braz, R.; Amir, E.; and Roth, D. 2005. Lifted first-order probabilistic inference. In *Proc. of IJCAI-05*, 1319–1325.