Supplementary Material for: Sampling and Inference for Beta Neutral-to-the-Left Models of Sparse Networks

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A DEGREE DISTIRBUTIONS WITH POWER LAW TAILS

Let D be a random variable from some distribution $P = (p_d)_{d\geq 1}$ with power law tails. Then

$$\mathbb{E}[D] = \sum_{d\geq 1} d \cdot p_d = C \cdot \sum_{d\geq d^*} d \cdot d^{-\eta} \ ,$$

for some constants C, d^* that control the tail approximation. The sum of terms $d^{-(\eta-1)}$ converges if and only if $\eta > 2$.

For a graph G_n, the average degree is

$$\bar{d}_n = \frac{1}{K_n} \sum_{j=1}^{K_n} d_{j,n} = \frac{2n}{K_n} \ . \tag{1}$$

Let D_n be the degree of a vertex sampled uniformly at random from G_n. If $m_d(n)/K_n \xrightarrow{\text{p}} p_d$ for all $d \in \mathbb{N}$, then $D_n \xrightarrow{\text{d}} D$ as $n \to \infty$ and $\bar{d}_n \xrightarrow{\text{a.s.}} \mathbb{E}[D]$.

If $K_n = o(n)$, then by (1) $\bar{d}_n \to \infty$, which implies that $\mathbb{E}[D] = \infty$. On the other hand, if $K_n = \Theta(n)$, then $\bar{d}_n \to \mathbb{E}[D] < \infty$.

The *Fact* in Section 2 is an assertion of these property.

B UNBOUNDED AVERAGE DEGREE IN EXCHANGEABLE POINT PROCESS MODELS

As with edge exchangeable models, models based on exchangeable point processes have unbounded expected average degree. We refer the reader to Caron and Fox (2017), Veitch and Roy (2015), and Borgs et al. (2016) for details on such models. Ignoring self-loops, the degree $D_\nu(\lambda)$ of a fixed vertex (with "position" $\lambda \in \mathbb{R}_+$) is Pois($\nu\mu_W(\lambda)$), where ν is the size parameter of the point process (Veitch and Roy, 2015, Lemma 5.1); taking $\nu \to \infty$ yields the asymptotic properties, and for non-trivial μ_W (i.e., those that generate sparse graphs), $\lim_{\nu\to\infty} \mathbb{E}[D_\nu(\lambda)] = \infty$ for all λ.

C ESTIMATORS FOR Ψ_j

When the arrival times are known, it is straightforward to show that the MLE for Ψ_j is

$$\hat{\Psi}_j^{\text{MLE}} = \frac{d_{j,n} - 1}{\bar{d}_{j,n} - T_j} \ . \tag{2}$$

If only the arrival order is observed, then the maximum a posteriori estimator (MAPE) corresponding to Ψ_j is

$$\hat{\Psi}_j^{\text{MAP}} = \frac{d_j - 1 - \alpha}{\bar{d}_j - j\alpha - 2} \ . \tag{3}$$

Note that the MAPE does not require knowledge of the arrival times, but requires specification of α. A consistent estimator that depends neither α nor the arrival times is given by (Bloem-Reddy and Orbanz, 2017)

$$\frac{d_j}{\bar{d}_j} \xrightarrow[n\to\infty]{\text{a.s.}} \Psi_j \quad \text{for all} \quad j \geq 1 \ . \tag{4}$$

D DETAILS OF MLEs FOR $\mathcal{PYP}$ AND GEOMETRIC INTERARRIVALS

Starting from equation (14) in the main text, the likelihood of observed data $\mathbf{Z}_n$ with degree distribution $m_n(d) := \#\{j : d_{j,n} = d\}$ under a BNTL model with inter-arrivals Geom(β) and NTL parameter α is

$$\mathbb{P}[\mathbf{Z}_n|\beta,\alpha] = \beta^{(K_n-1)}(1-\beta)^{(n-K_n-1)} \times \prod_{i\notin \mathbf{T}_{K_n}} \frac{d_{z_i,i} - \alpha}{i - 1 - K_{i-1}\alpha} \tag{5}$$

Observe that

$$\prod_{i \notin \mathbf{T}_{K_n}} (d_{z_i,i} - \alpha) = \prod_{j=1}^{K_n} \prod_{i=2}^{d_{j,n}} (i - 1 - \alpha)$$
$$= \prod_{j=1}^{K_n} \frac{\Gamma(d_{j,n} - \alpha)}{\Gamma(1-\alpha)}$$
$$= \Gamma(1-\alpha)^{-K_n} \prod_{d=1}^{\infty} \Gamma(d-\alpha)^{m_n(d)} \tag{6}$$

This yields

$$\begin{aligned}\mathbb{P}[\mathbf{Z}_n|\beta,\alpha] = \quad & \beta^{(K_n-1)}(1-\beta)^{(n-K_n-1)} \\ & \times \Gamma(1-\alpha)^{-K_n} \prod_{d=1}^{\infty} \Gamma(d-\alpha)^{m_n(d)} \\ & \times \prod_{i \notin \{T_j\}} (i - 1 - K_{i-1}\alpha)^{-1} \end{aligned} \tag{7}$$

For the coupled $\mathcal{PYP}$, the BNTL parameter $\alpha = \tau$ is coupled to the arrival process. The factorization (14) is less helpful. The full likelihood in this case is

$$\begin{aligned}\mathbb{P}[\mathbf{Z}_n|\theta,\tau] = \quad & \prod_{j=1}^{K_n} \frac{\theta + j\tau}{T_j - 1 + \theta} \\ & \times \prod_{i \notin \mathbf{T}_{K_n}} \frac{d_{z_i,i} - \tau}{i - 1 - \theta} \end{aligned} \tag{8}$$

$$\begin{aligned} = \quad & \frac{\Gamma(1+\theta)}{\Gamma(n+\theta)} \prod_{j=1}^{K_n} (\theta + j\tau) \\ & \times \Gamma(1-\tau)^{-K_n} \prod_{d=1}^{\infty} \Gamma(d-\tau)^{m_n(d)} \end{aligned} \tag{9}$$

Finally, for the uncoupled $\mathcal{PYP}$ in which α and τ are independent parameters, we again make use of (14) to write the likelihood as

$$\begin{aligned}\mathbb{P}[\mathbf{Z}_n|\beta,\alpha] = \quad & \frac{\Gamma(1+\theta)}{\Gamma(n+\theta)} \prod_{j=1}^{K_n} (\theta + j\tau) \\ & \times \prod_{i \notin \mathbf{T}_{K_n}} (i - 1 - K_{i-1}\tau) \\ & \times \Gamma(1-\alpha)^{-K_n} \prod_{d=1}^{\infty} \Gamma(d-\alpha)^{m_n(d)} \\ & \times \prod_{i \notin \mathbf{T}_{K_n}} (i - 1 - K_{i-1}\alpha)^{-1} \end{aligned} \tag{10}$$

and one can readily see that setting $\alpha = \tau$ reduces to (9).

References

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