$$L(x,\lambda) = \mathbb{E}[R(N)] - \sum_{i\in\mathcal{U}} p_i^s x_i \mathbb{E}[T_N] - \sum_{i\in\mathcal{U}} \lambda_i(\frac{1-\sum_j p_j^s F_R(x_j)}{p_i^s[1-F_R(x_i)]}\mathbb{E}[T_N] - \nu_i)$$
$$= \mathbb{E}[R(N)] - \sum_{i\in\mathcal{U}} (p_i^s x_i + \lambda_i \frac{1-\sum_j p_j^s F_R(x_j)}{p_i^s[1-F_R(x_i)]})\mathbb{E}[T_N] + \sum_{i\in\mathcal{U}} \lambda_i \nu_i \;. \tag{13}$$

$$0 = -\lambda_i^* \nu_i + \lambda_i^* \frac{1-\sum_j p_j^s F_R(x_j)}{p_i^s[1-F_R(x_i)]} \cdot \Big[\frac{\tau}{1-F_R(\sum_i (p_i^s x_i + \lambda_i^* \frac{1-\sum_j p_j^s F_R(x_j)}{p_i^s[1-F_R(x_i)]}) + V^*)} + 1\Big] \;. \tag{14}$$
$$\Rightarrow p_i^s[1-F_R(x_i)]\nu_i = (1-\sum_j p_j^s F_R(x_j)) \cdot \Big[\frac{\tau}{1-F_R(\sum_i (p_i^s x_i + \lambda_i^* \frac{1-\sum_j p_j^s F_R(x_j)}{p_i^s[1-F_R(x_i)]}) + V^*)} + 1\Big] \;. \tag{15}$$

Supplementary materials: Proof of Theorem 4.3

To solve the constrained optimization problem, we introduce its corresponding Lagrange relaxation in Eqn.(13). Denote

$$V^* := \mathbb{E}[R(N)] - \sum_i (p_i^s x_i + \lambda_i \frac{1-\sum_j p_j^s F_R(x_j)}{p_i^s[1-F_R(x_i)]})\mathbb{E}[T_N] \;.$$

Following previous work [Ferguson, 2006] we know the optimal stopping time for maximizing V^* is given by

$$N^* = \min\{n \geq 1 : R(n) \geq \sum_i (p_i^s x_i + \lambda_i \frac{1-\sum_j p_j^s F_R(x_j)}{p_i^s[1-F_R(x_i)]}) + V^*\} \;,$$

and the optimal set of thresholds satisfy the following fixed-point equation:

$$\mathbb{E}\Big[R(N) - \sum_i (p_i^s x_i + \lambda_i \frac{1-\sum_j p_j^s F_R(x_j)}{p_i^s[1-F_R(x_i)]}) - V^*\Big]^+ = \sum_i (p_i^s x_i + \lambda_i \frac{1-\sum_j p_j^s F_R(x_j)}{p_i^s[1-F_R(x_i)]})\tau \;. \tag{16}$$

Again following the results of [Ferguson, 2006], we are able to characterize the expected stopping time as follows,

$$\mathbb{E}[T_N] = \frac{\tau}{1-F_R(\sum_i (p_i^s x_i + \lambda_i \frac{1-\sum_j p_j^s F_R(x_j)}{p_i^s[1-F_R(x_i)]}) + V^*)} + 1 \;. \tag{17}$$

Consider the dual problem $L(\mathbf{x},\lambda) = V^* + \lambda^{\mathbf{T}}\nu$. By complementary slackness we have:

$$\lambda_i^* \frac{1-\sum_{j\in\mathcal{U}} p_j^s F_R(x_j)}{p_i^s[1-F_R(x_i)]}\mathbb{E}[T_N] - \lambda_i^* \nu_i = 0, \forall i \in \mathcal{U}. \tag{18}$$

Substitute (17) into (18) we have $\forall i \in \mathcal{U}$, we have Eqn.(14).

We first consider active constraints, i.e., constraints with $\lambda_i^* > 0$. (recall in the dual problem we have the variables $\lambda_i^* \geq 0, \forall i \in \mathcal{U}$.) For these active constraints we will get Eqn.(15). Notice the RHS of above equation does not depend on any specific index i. Therefore we have

$$p_i^s[1-F_R(x_i)]\nu_i = C(\mathbf{x},\lambda^*) \;,$$

where $C(\mathbf{x},\lambda^*)$ is a constant that is independent of indices. Moreover for $\lambda_i^*, \lambda_j^* > 0, i \neq j$ we have

$$\frac{1-F_R(x_i)}{1-F_R(x_j)} = \frac{p_j^s \nu_j}{p_i^s \nu_i} \;.$$

Denote $\nu(\mathbf{x}) := \frac{p_i^s[1-F_R(x_i)]\nu_i}{1-\sum_j p_j^s F_R(x_j)}$, we will get

$$\sum_i \left(p_i^s x_i + \lambda_i^* \frac{1-\sum_j p_j^s F_R(x_j)}{p_i^s[1-F_R(x_i)]} \right) + V^* = F_X^{-1}(1 - \frac{\tau}{\nu(\mathbf{x})-1}) \ .$$

Plug above to Eqn.(16), the following establishes

$$\mathbb{E}\Big[R(N) - F_R^{-1}(1 - \frac{\tau}{\nu(\mathbf{x})-1})\Big]^+ = \sum_i (p_i^s x_i + \lambda_i^* \frac{1-\sum_j p_j^s F_R(x_j)}{p_i^s[1-F_R(x_i)]})\tau \ ,$$

and

$$V^* = F_R^{-1}(1 - \frac{\tau}{\nu(\mathbf{x})-1}) - \frac{1}{\tau} \cdot \mathbb{E}\Big[R(N) - F_R^{-1}(1 - \frac{\tau}{\nu(\mathbf{x})-1})\Big]^+ \ .$$

Also for our primal and dual problems we have

$$0 = L(\mathbf{x}^*, \lambda^*) = V^* + \lambda^{*\mathbf{T}}\nu \ .$$

Consider the $\lambda^{*\mathbf{T}}\nu$ term. Notice

$$\begin{aligned}
\lambda_i^* \nu_i &= \lambda_i^* \frac{1-\sum_j p_j^s F_R(x_j)}{p_i^s[1-F_R(x_i)]} \cdot \frac{p_i^s[1-F_R(x_i)]}{1-\sum_j p_j^s F_R(x_j)} \nu_i \\
&= \nu(\mathbf{x}) \cdot \lambda_i^* \frac{1-\sum_j p_j^s F_R(x_j)}{p_i^s[1-F_R(x_i)]} \ .
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\lambda^{*\mathbf{T}}\nu &= \nu(\mathbf{x}) \cdot \sum_i \lambda_i^* \frac{1-\sum_j p_j^s F_R(x_j)}{p_i^s[1-F_R(x_i)]} \\
&= \nu(\mathbf{x}) \cdot \Big[\frac{\mathbb{E}[R(N) - F_R^{-1}(1 - \frac{\tau}{\nu(\mathbf{x})-1})]^+}{\tau} - \sum_i p_i^s x_i \Big] \ .
\end{aligned}$$

Moreover we have Eqn.(19):

$$\begin{aligned}
& F_R^{-1}(1 - \frac{\tau}{\nu(\mathbf{x})-1}) - \frac{1}{\tau} \cdot \mathbb{E}\Big[R(N) - F_R^{-1}(1 - \frac{\tau}{\nu(\mathbf{x})-1})\Big]^+ \\
& + \nu(\mathbf{x}) \cdot \left\{ \frac{\mathbb{E}\left[R(N) - F_R^{-1}(1 - \frac{\tau}{\nu(\mathbf{x})-1})\right]^+}{\tau} - \sum_i p_i^s x_i \right\} = 0 \ .
\end{aligned} \tag{19}$$

Notice the following holds:

$$\nu(\mathbf{x}) = \frac{p_i^s[1-F_R(x_i)]\nu_i}{1-\sum_j p_j^s F_R(x_j)} = \frac{p_i^s[1-F_R(x_i)]\nu_i}{\sum_j p_j^s(1-F_R(x_j))} \ ,$$

where the second equality is due to $\sum_j p_j^s = 1$. Substitute in

$$1 - F_R(x_j) = \frac{p_i^s \nu_i}{p_j^s \nu_j}(1 - F_R(x_i))$$

we have

$$\nu(\mathbf{x}) = \frac{p_i^s[1-F_R(x_i)]\nu_i}{\sum_j p_j^s \frac{p_i^s\nu_i}{p_j^s\nu_j}(1-F_R(x_i))} = \frac{1}{\sum_{j\in\mathcal{U}}\frac{1}{\nu_j}} = \nu_{\mathcal{U}} ,$$

which is a constant. From which we have

$$F_R^{-1}(1-\frac{\tau}{\nu(\mathbf{x})-1}) = C_1 ,$$

$$\frac{1}{\tau}\cdot\mathbb{E}[R(N)-F_R^{-1}(1-\frac{\tau}{\nu(\mathbf{x})-1})]^+ = C_2 ,$$

with C_1, C_2 being some constant. Notice $C_1 - C_2$ is our previously defined function g evaluated at $\nu(\mathbf{x})$. Therefore we have

$$C_1 - C_2 + \nu_{\mathcal{U}}(C_2 - \sum_i p_i^s x_i) = 0 .$$

Theoretically with the other equations $\frac{1-F_R(x_i)}{1-F_R(x_j)} = \frac{p_j^s\nu_j}{p_i^s\nu_i}$ we can solve $\mathbf{x}^*$. The above equation can be reduced to

$$\sum_{i\in\mathcal{U}} p_i^s x_i = x_{\mathcal{U}}^*.$$

The rest is to check whether the corresponding λ^*s are positive. This is equivalent with deciding whether the current constraints are active or not. In fact from previous derivations we know

$$\lambda^T\nu = \frac{\mathbb{E}[R(N)-F_R^{-1}(1-\frac{\tau}{\nu_{\mathcal{U}}-1})]^+}{\tau} - \sum_i p_i^s x_i^*$$

therefore we could derive a positive set of λ if the following holds,

$$\frac{\mathbb{E}[R(N)-F_R^{-1}(1-\frac{\tau}{\nu_{\mathcal{U}}-1})]^+}{\tau} - \sum_i p_i^s x_i^* > 0$$

which further gives us

$$\frac{1}{\tau}\mathbb{E}[R(N)-F_R^{-1}(1-\frac{\tau}{\nu_{\mathcal{U}}-1})]^+ - F_R^{-1}(1-\frac{\tau}{\nu_{\mathcal{U}}-1}) > 0 ,$$

since ν^* is the solution for above equation when equality holds, and by simple algebra we could show for $\nu_{\mathcal{U}} < \nu^*$ the above condition holds.

Now we show under what condition the solutions $\mathbf{x}^*$ is non-negative. Note from

$$\frac{1-F_R(x_i)}{1-F_R(x_j)} = \frac{p_j^s\nu_j}{p_i^s\nu_i} ,$$

we know

$$x_i = F_R^{-1}\left(1-[1-F_R(x_j)]\frac{p_j^s\nu_j}{p_i^s\nu_i}\right) .$$

Not hard to see the smallest x_j associated with the minimum $p_j^s\nu_j$. Therefore

$$\sum_{i\in\mathcal{U}\setminus j} F_R^{-1}\left(1-[1-F_R(x_j)]\frac{\min_{j\in\mathcal{U}} p_j^s\nu_j}{p_i^s\nu_i}\right) + x_j = x_{\mathcal{U}}^* .$$

In order to make sure $\mathbf{x}^*$ is non-negative, we need $x_j \geq 0$, i.e.,

$$\sum_{i\in\mathcal{U}} F_R^{-1}\left(1-\frac{\min_{j\in\mathcal{U}} p_j^s\nu_j}{p_i^s\nu_i}\right) \leq x_{\mathcal{U}}^* .$$