Appendix A Preliminaries

We will extensively use the following two facts from probability theory. The first fact is a well known Hoeffding's inequality, and the second fact connects the cdfs of Beta and Binomial random variables.

Fact 1. ***Hoeffding's inequality (Hoeffding, 1963)*** *Let $X_1, \dots, X_n$ be independent and bounded random variables with $X_i \in [a, b]$ for all i, where $-\infty < a \leq b < \infty$. Then,*

$$\mathbb{P}\left(\sum_{i=1}^{n} |X_i - \mathbb{E}[X_i]| \geq t\right) \leq 2\exp\left\{-\frac{2t^2}{n(b-a)^2}\right\}.$$

We have the following simple observation using Fact 1.

Observation 1. *For all $n, p \in [0, 1], \delta \geq 0$,*

$$F^{B}_{n+1,p}(np - n\delta) \leq e^{-2n\delta^2}.$$

Proof.

$$\begin{aligned} F^{\text{B}}_{n+1,p}(r) &= (1-p)F^{\text{B}}_{n,p}(r) + pF^{\text{B}}_{n,p}(r-1) \\ &\leq (1-p)F^{\text{B}}_{n,p}(r) + pF^{\text{B}}_{n,p}(r) \\ &= F^{\text{B}}_{n,p}(r). \end{aligned}$$

Using Fact 1 we have $F^{\text{B}}_{n,p}(np - n\delta) \leq e^{-2n\delta^2}$. Combining that with the above identity gives us the proof. □

Fact 2. ***Beta-Binomial link (Agrawal and Goyal, 2012)*** *Let $F^{Beta}_{a,b}$ be the cdf of a Beta distribution with parameters a and b, and $F^{B}_{n,p}$ be the cdf of a Binomial distribution with parameters n and p. Then,*

$$F^{Beta}_{\alpha,\beta}(y) = 1 - F^{B}_{\alpha+\beta-1,y}(\alpha - 1).$$

Appendix B Omitted Proofs

B.1 Proof of Lemma 2

Lemma 2. *Let $i, j \in [K]$ be a pair of arms such that $i < j$, and let $L_{i,j} = 32/\Delta^2_{i,j}$. Then, at any time $t \leq T$,*

$$\mathbb{P}(\theta_{j,t} \geq \mu_j + \Delta_{i,j}/2, n_{j,t} \geq L_{i,j}\ln(a_{j,T})) \leq 2a^{-3}_{j,T}.$$

Proof. For simplicity of notation, let $\delta_{i,j} = \Delta_{i,j}/2$. We also introduce $\hat{\mu}_{i,t}$ as the empirical mean of arm i until time t. We use the fact that $\hat{\mu}_{i,t} = s_{i,t}/n_{i,t}$ in the following proof.

$$\begin{aligned} &\mathbb{P}(\theta_{j,t} \geq \mu_j + \delta_{i,j}, n_{j,t} \geq L_{i,j}\ln(a_{j,T})) \\ \leq\; &\underbrace{\mathbb{P}\left(\theta_{j,t} \geq \mu_j + \delta_{i,j}, \hat{\mu}_{j,t} \leq \mu_j + \frac{\delta_{i,j}}{2}, n_{j,t} \geq L_{i,j}\ln(a_{j,T})\right)}_{A} \\ &+ \underbrace{\mathbb{P}\left(\hat{\mu}_{j,t} > \mu_j + \frac{\delta_{i,j}}{2}, n_{j,t} \geq L_{i,j}\ln(a_{j,T})\right)}_{B} \end{aligned}$$

Bounding term A:

$$\begin{aligned} &\mathbb{P}\left(\theta_{j,t} \geq \mu_j + \delta_{i,j}, \hat{\mu}_{j,t} \leq \mu_i + \frac{\delta_{i,j}}{2}, n_{j,t} \geq L_{i,j}\ln(a_{j,T})\right) \\ \leq\; &\sum_{l=L_{i,j}\ln(a_{j,T})}^{a_{j,T}} \mathbb{P}\left(\theta_{j,t} \geq \hat{\mu}_{j,t} + \frac{\delta_{i,j}}{2}, n_{j,t} = l\right) \\ =\; &\sum_{l=L_{i,j}\ln(a_{j,T})}^{a_{j,T}} \left(1 - F^{\text{Beta}}_{s_{i,t}+1, l-s_{i,t}+1}\left(\hat{\mu}_{j,t} + \frac{\delta_{i,j}}{2}\right)\right) \\ =\; &\sum_{l=L_{i,j}\ln(a_{j,T})}^{a_{j,T}} F^{\text{B}}_{n_{j,t}+1, \hat{\mu}_{j,t}+\frac{\delta_{i,j}}{2}}(s_{i,t}) && \text{(Using Fact 2)} \\ \leq\; &\sum_{l=L_{i,j}\ln(a_{j,T})}^{a_{j,T}} \exp\left\{-\frac{l\delta^2_{i,j}}{2}\right\} && \text{(Using Observation 1)} \\ \leq\; &a_{j,T}\exp\left\{-\frac{L_{i,j}\ln(a_{j,T})\delta^2_{i,j}}{2}\right\} \\ =\; &a_{j,T}e^{-4\ln(a_{j,T})} = a^{-3}_{j,T} \end{aligned}$$

Bounding term B:

$$\begin{aligned} &\mathbb{P}\left(\hat{\mu}_{j,t} > \mu_i + \frac{\delta_{i,j}}{2}, n_{i,t} \geq L_{i,j}\ln(a_{j,T})\right) \\ =\; &\sum_{l=L_{i,j}\ln(a_{j,T})}^{a_{j,T}} \mathbb{P}\left(\hat{\mu}_{j,t} > \mu_i + \frac{\delta_{i,j}}{2}, n_{i,t} = l\right) \\ \leq\; &\sum_{l=L_{i,j}\ln(a_{j,T})}^{a_{j,T}} \exp\left\{-\frac{l\delta^2_{i,j}}{2}\right\} && \text{(Using Fact 1)} \\ \leq\; &a_{j,T}\exp\left\{-\frac{L_{i,j}\ln(a_{j,T})\delta^2_{i,j}}{2}\right\} \\ \leq\; &a_{j,T}e^{-4\ln(a_{j,T})} \leq a^{-3}_{j,T} \end{aligned}$$

□

B.2 Proof of Lemma 3

Lemma 3. *Let $e_{i,t} = \sqrt{\frac{4\ln(w_{i,t})}{n_{i,t}}}$ for any arm i. Then, at any time t, we have,*

$$\mathbb{P}(\theta_{i,t} \leq \mu_i - e_{i,t}) \leq w^{-2}_{i,t},$$

where $w_{i,t}$ denotes the number of times arm i is the best available arm until time t.

Proof.

$$\mathbb{P}\big(\theta_{i,t} \le \mu_i - e_{i,t}\big) = F^{\text{Beta}}_{s_{i,t}+1, n_{i,t}-s_{i,t}+1}(\mu_i - e_{i,t})$$
$$= 1 - F^{\text{B}}_{n_{i,t}+1, \mu_i - e_{i,t}}(s_{i,t}) \quad \text{(Using Fact 2)}$$

Let $X_{1,l} \sim \text{Bernoulli}(\mu_i - e_{i,t})$ and $X_{2,l} \sim \text{Bernoulli}(\mu_i)$ be Bernoulli r.v.s. and let $Z_l = X_{2,l} - X_{1,l}$ be a discreet r.v. with values in $\{-1, 0, 1\}$ and mean $e_{i,t}$. Then,

$$\begin{aligned}
&= 1 - \mathbb{P}\left(\sum_{l=1}^{n_{i,t}+1} X_{1,l} \le \sum_{l=1}^{n_{i,t}} X_{2,l}\right) \\
&= \mathbb{P}\left(\sum_{l=1}^{n_{i,t}} X_{2,l} < \sum_{l=1}^{n_{i,t}+1} X_{1,l}\right) \\
&= \mathbb{P}\left(\sum_{l=1}^{n_{i,t}} Z_l < 1\right) \\
&= \mathbb{P}\left(\sum_{l=1}^{n_{i,t}} Z_l \le 0\right) \qquad \text{(As } X_{1,l} \le 1\text{)} \\
&= \mathbb{P}\left(\sum_{l=1}^{n_{i,t}} (Z_l - e_{i,t}) \le -n_{i,t} e_{i,t}\right) \\
&\le \exp\left\{-\frac{2 n_{i,t}^2 e_{i,t}^2}{4 n_{i,t}}\right\} \qquad \text{(Using Fact 1)} \\
&= e^{-2\ln(w_{i,t})} = w_{i,t}^{-2}
\end{aligned}$$

□

B.3 Proof of Lemma 5

Lemma 5. *For any fixed arm $a > i$,*

$$\mathbb{P}\left(\exists\, t' \in \mathcal{I}^{(i)}_{j,l} : \theta_{a,t'} \ge \mu_a + \delta_{i,a}, n^{(i)}_{a,t'} \ge L_{i,a}\ln(w_{i,t})\right) \le 2 w_{i,t}^{-2+b}.$$

Proof.

$$\mathbb{P}\left(\theta_{a,t'} \ge \mu_a + \delta_{i,a}, n^{(i)}_{a,t'} \ge L_{i,a}\ln(w_{i,t})\right) \le \underbrace{\mathbb{P}\left(\theta_{a,t'} \ge \mu_a + \delta_{i,a}, \hat{\mu}_{a,t'} \le \mu_a + \frac{\delta_{i,a}}{2}, n^{(i)}_{a,t'} \ge L_{i,a}\ln(w_{i,t})\right)}_{A} + \underbrace{\mathbb{P}\left(\hat{\mu}_{a,t'} > \mu_a + \frac{\delta_{i,a}}{2}, n^{(i)}_{a,t'} \ge L_{i,a}\ln(w_{i,t})\right)}_{B}$$

Bounding term A:

$$\begin{aligned}
&\mathbb{P}\left(\theta_{a,t'} \ge \mu_a + \delta_{i,a}, \hat{\mu}_{a,t'} \le \mu_a + \frac{\delta_{i,a}}{2}, n^{(i)}_{a,t'} \ge L_{i,a}\ln(w_{i,t})\right) \\
&\le \sum_{l=L_{i,a}\ln(w_{i,t})}^{w_{i,t}} \mathbb{P}\left(\theta_{a,t'} \ge \hat{\mu}_{a,t'} + \frac{\delta_{i,a}}{2}, n^{(i)}_{a,t'} = l\right) \\
&= \sum_{l=L_{i,a}\ln(w_{i,t})}^{w_{i,t}} \left(1 - F^{\text{Beta}}_{s_{a,t'}+1, n_{a,t'}-s_{a,t'}+1}\left(\hat{\mu}_{a,t'} + \frac{\delta_{i,a}}{2}\right)\right) \\
&= \sum_{l=L_{i,a}\ln(w_{i,t})^{w_{i,t}}} F^{\text{B}}_{n_{a,t'}+1, \hat{\mu}_{a,t'} + \frac{\delta_{i,a}}{2}}(s_{a,t'}) \qquad \text{(Using Fact 2)} \\
&\le \sum_{l=L_{i,a}\ln(w_{i,t})}^{w_{i,t}} \exp\left\{-\frac{n_{a,t'}\delta_{i,a}^2}{2}\right\} \qquad \text{(Using Observation 1)} \\
&\le w_{i,t}\exp\left\{-\frac{L_{i,a}\ln(w_{i,t})\delta_{i,a}^2}{2}\right\} \\
&= w_{i,t} e^{-4\ln(w_{i,t})} = w_{i,t}^{-3}
\end{aligned}$$

Bounding term B:

$$\begin{aligned}
&\mathbb{P}\left(\hat{\mu}_{a,t'} > \mu_a + \frac{\delta_{i,a}}{2}, n^{(i)}_{a,t'} \ge L_{i,a}\ln(w_{i,t})\right) \\
&= \sum_{l=L_{i,a}\ln(w_{i,t})}^{w_{i,t}} \mathbb{P}\left(\hat{\mu}_{a,t'} > \mu_a + \frac{\delta_{i,a}}{2}, n^{(i)}_{a,t'} = l\right) \\
&\le \sum_{l=L_{i,a}\ln(w_{i,t})}^{w_{i,t}} \exp\left\{-\frac{n_{a,t'}\delta_{i,a}^2}{2}\right\} \qquad \text{(Using Fact 1)} \\
&\le w_{i,t}\exp\left\{-\frac{L_{i,a}\ln(w_{i,t})\delta_{i,a}^2}{2}\right\} \\
&\le w_{i,t} e^{-4\ln(w_{i,t})} \le w_{i,t}^{-3}
\end{aligned}$$

Thus we get,

$$\mathbb{P}\left(\theta_{a,t'} \ge \mu_a + \delta_{i,a}, n^{(i)}_{a,t'} \ge L_{i,a}\ln(w_{i,t})\right) \le 2 w_{i,t}^{-3}$$

Now,

$$\begin{aligned}
&\mathbb{P}\left(\exists t' \in \mathcal{I}^{(i)}_{j,l}, a > i : \theta_{a,t'} > \mu_a + \delta_{i,a}, n^{(i)}_{a,t'} > L_{i,a}\ln(w_{i,t})\right) \\
&\le \sum_{t' \in \mathcal{I}^{(i)}_{j,l}} \sum_{a>i} \mathbb{P}\left(\theta_{a,t'} \ge \mu_a + \delta_{i,a}, n^{(i)}_{a,t'} \ge L_{i,a}\ln(w_{i,t})\right) \\
&\le \sum_{t' \in \mathcal{I}^{(i)}_{j,l}} \sum_{a>i} 2 w_{i,t}^{-3} \\
&\le \frac{w_{i,t}^{1-b}}{K-i} \cdot (K-i) \cdot 2 w_{i,t}^{-3} = 2 w_{i,t}^{-(2+b)}
\end{aligned}$$

□

B.4 Proof of Lemma 6

Lemma 6. *There exists a constant $\lambda_0 > 1$ such that for all $\lambda \in (1, \lambda_0)$, for any interval $\mathcal{J}^{(i)} \subseteq \cup_{l \in [K-i+1]} \mathcal{I}^{(i)}_{j,l}$ and for every positive function f, we have*

$$\mathbb{P}\left(\{\forall s \in \mathcal{J}^{(i)} : \theta_{i,s} \le y_i\} \cap \{\left|\mathcal{J}^{(i)}\right| \ge f(t)\}\right) \le (\alpha_i)^{f(t)} + C_{\lambda,i}\frac{1}{f(t)^\lambda} e^{-j d_{\lambda,i}},$$

where $C_{\lambda,i} > 0$, $d_{\lambda,i} > 0$, $\alpha_i = \left(\frac{1}{2}\right)^{1-\mu_{i+1}-\delta_i}$ and $y_i = \mu_{i+1} + \delta_{i,i+1}$ for every $i \in \{1, \ldots, K-1\}$.

Proof. Recall that $\tau_j^{(i)}$ denotes the time of occurrence of the j^{th} pull of arm i and $\xi_j^{(i)}$ denotes the i-optimal timeline between the j^{th} and $(j+1)^{th}$ pull of arm i. Hence, on the interval $\mathcal{J}^{(i)}$ (included in $\xi_j^{(i)}$ by hypothesis) arm i has a fixed posterior Beta$(s_{i,\tau_j^{(i)}}+1, j-s_{i,\tau_j^{(i)}}+1)$ distribution when conditioned on $s_{i,\tau_j^{(i)}}$. Thus,

$$\mathbb{P}\Big(\theta_{i,s} \le y_i \mid s \in \mathcal{J}^{(i)}\Big) \le F^{\text{Beta}}_{s_i+1,j-s_i+1}(y_i) \quad \text{(Using } s_i = s_{i,\tau_j^{(i)}}\text{)}$$
$$= 1 - F^{\text{B}}_{j+1,y_i}(s_i) \quad \text{(Using Fact 2)}$$

This gives us,

$$\begin{aligned}
&\mathbb{P}\Big(\forall s \in \mathcal{J}^{(i)}, \theta_{i,s} \le y_i\Big) \\
&= \mathbb{E}\Big[\mathbb{P}\Big(\forall s \in \mathcal{J}^{(i)}, \theta_{i,s} \le y_i \mid s_{i,\tau_j^{(i)}}\Big)\Big] \\
&\le \mathbb{E}\Bigg[\Big(1 - F^{\text{B}}_{j+1,y_i}\Big(s_{i,\tau_j^{(i)}}\Big)\Big)^{|\mathcal{J}^{(i)}|}\Bigg] \\
&\le \mathbb{E}\Bigg[\Big(1 - F^{\text{B}}_{j+1,y_i}\Big(s_{i,\tau_j^{(i)}}\Big)\Big)^{f(t)}\Bigg] \\
&= \sum_{s=0}^{j}\Big(1 - F^{\text{B}}_{j+1,y_i}\Big(s_{i,\tau_j^{(i)}}\Big)\Big)^{f(t)} f^{\text{B}}_{j,\mu_i}(s) \\
&\le \sum_{s=0}^{j}\exp\Big(-f(t)(1-y)F^{\text{B}}_{j+1,y}(s)\Big) f^{\text{B}}_{j,\mu_i}(s)
\end{aligned}$$

Using $F^{\text{B}}_{j+1,y_i}(s) = (1-y_i)F^{\text{B}}_{j,y_i}(s) + y_i F^{\text{B}}_{j,y_i}(s-1) \ge (1-y_i)F^{\text{B}}_{j+1,y_i}(s)$ we get:

$$\begin{aligned}
&\le \sum_{s=0}^{j}\exp\Big(-f(t)(1-y_i)F^{\text{B}}_{j,y}(s)\Big) f^{\text{B}}_{j,\mu_i}(s) \\
&\le \sum_{s=0}^{\lfloor y_i j\rfloor}\exp\Big(-f(t)(1-y_i)F^{\text{B}}_{j,y_i}(s)\Big) f^{\text{B}}_{j,\mu_i}(s) \\
&\quad + \sum_{s=\lceil y_i j\rceil}^{j}\left(\frac{1}{2}\right)^{(1-y_i)f(t)} f^{\text{B}}_{j,\mu_i}(s) \\
&\le \underbrace{\sum_{s=0}^{\lfloor y_i j\rfloor}\exp\Big(-f(t)(1-y_i)F^{\text{B}}_{j,y_i}(s)\Big) f^{\text{B}}_{j,\mu_i}(s)}_{E} \\
&\quad + \sum_{s=\lceil y_i j\rceil}^{j}\left(\frac{1}{2}\right)^{(1-y_i)f(t)}
\end{aligned}$$

For every $\lambda > 1, \forall x > 0, x^\lambda \exp(-x) \le \left(\frac{\lambda}{e}\right)^\lambda$. Taking $C_\lambda = \left(\frac{\lambda}{e}\right)^\lambda$, we get the following upper-bound:

$$(E) \le \frac{C_\lambda}{(f(t)(1-y_i))^\lambda}\sum_{s=0}^{\lfloor y_i j\rfloor}\frac{f^{\text{B}}_{j,\mu_i}(s)}{\big(F^{\text{B}}_{j,y_i}(s)\big)^\lambda}$$
$$\le \frac{C_\lambda}{(f(t)(1-y_i))^\lambda}\sum_{s=0}^{\lfloor y_i j\rfloor}\frac{f^{\text{B}}_{j,\mu_i}(s)}{\big(f^{\text{B}}_{j,y_i}(s)\big)^\lambda}$$

Now,

$$\begin{aligned}
\frac{f^{\text{B}}_{j,\mu_i}(s)}{\big(f^{\text{B}}_{j,y_i}(s)\big)^\lambda} &= \frac{\binom{j}{s}\mu_i^s(1-\mu_i)^{j-s}}{\binom{j}{s}^\lambda (y^\lambda)^s((1-y)^\lambda)^{j-s}} \\
&\le \frac{\mu_i^s(1-\mu_i)^{j-s}}{(y^\lambda)^s((1-y)^\lambda)^{j-s}} \\
&\le \left(\frac{1-\mu_i}{(1-y)^\lambda}\right)^j\left(\frac{\mu_i(1-y)^\lambda}{y^\lambda(1-\mu_i)}\right)^s
\end{aligned}$$

Let $R_\lambda(\mu_i, y_i) = \frac{\mu_i(1-y_i)^\lambda}{y^\lambda(1-\mu_i)}$. There exists some $\lambda_1 > 1$ such that, if $\lambda < \lambda_1, R_\lambda > 1$. More precisely,

$$\lambda_1(\mu_i, y_i) = \begin{cases} \dfrac{\ln\left(\frac{\mu_i}{1-\mu_i}\right)}{\ln\left(\frac{y_i}{1-y_i}\right)} & \text{, if } y > \frac{1}{2} \\ +\infty & \text{, if } y < \frac{1}{2}\end{cases}$$

For $1 < \lambda < \lambda_1$:

$$\begin{aligned}
\sum_{s=0}^{\lfloor y_i j\rfloor}\frac{f^{\text{B}}_{j,\mu_i}(s)}{\big(f^{\text{B}}_{j,y_i}(s)\big)^\lambda} &\le \left(\frac{1-\mu_i}{(1-y)^\lambda}\right)^j\sum_{s=0}^{\lfloor y_i j\rfloor}R_\lambda^s \\
&= \left(\frac{1-\mu_i}{(1-y)^\lambda}\right)^j\frac{R_\lambda^{\lfloor y_i j\rfloor+1}}{R_\lambda - 1} \\
&\le \left(\frac{1-\mu_i}{(1-y)^\lambda}\right)^j\frac{R_\lambda}{R_\lambda - 1}R_\lambda^{y_i j} \\
&= \frac{R_\lambda}{R_\lambda - 1}\left(\frac{1-\mu_i}{(1-y)^\lambda}\right)^{j-y_i j}\left(\frac{\mu_i}{y^\lambda}\right)^{y_i j} \\
&= \frac{R_\lambda}{R_\lambda - 1}e^{-jd_\lambda(y_i,\mu_i)}
\end{aligned}$$

where,

$$\begin{aligned}
d_\lambda(y_i, \mu_i) &= y_i\ln\left(\frac{y_i^\lambda}{\mu_i}\right) + (1-y_i)\ln\left(\frac{(1-y_i)^\lambda}{1-\mu_i}\right) \\
&= \lambda[y_i\ln(y_i) + (1-y_i)\ln(1-y_i)] \\
&\quad - [y_i\ln(\mu_i) + (1-y_i)\ln(1-\mu_i)]
\end{aligned}$$

which is an affine function of λ with negative slope $(y_i\ln(y_i) + (1-y_i)\ln(1-y_i) < 0)$ for all $y \in (0,1)$ and $d_1(y_i, \mu_i) = KL(y_i, \mu_i)$. Hence, for fixed $0 < y_i < \mu_i \le 1$ this function is positive whenever

$$\lambda < \frac{y_i\ln(\mu_i) + (1-y_i)\ln(1-\mu_i)}{y_i\ln(y_i) + (1-y_i)\ln(1-y_i)} =: \lambda_2(\mu_i, y_i)$$

Clearly, $\lambda_2(\mu_i, y_i) > 1$ and we know from Kaufmann et al. (2012) that $\lambda_2 \le \lambda_1$. So we choose $\lambda_0(\mu_i, y_i) = \lambda_2(\mu_i, y_i)$.

To conclude the proof we define $d_{\lambda,i} := d_\lambda(y_i, \mu_i)$ and $C_{\lambda,i} := C_\lambda(1-y_i)^{-\lambda}\frac{R_\lambda}{1-R_\lambda}$. □