A HELGASON TRANSFORMS

The reader is referred to Terras (1985) for a general defintion of the Helgason-Fourier transform on a symmetric space. We specialize those constructions for $\mathbb{H}_2$, regarded in this section as the Poincaré half-plane with the metric

$$dz = {}^{dx\,dy}/_{y^2}.$$

Let f and ϕ denote smooth maps

$$f : \mathbb{H}_2 \to \mathbb{C}, \quad \phi : \mathbb{R} \times \mathbb{SO}_2 \to \mathbb{C}$$

with compact supports.

The *Helgason-Fourier transform* $\mathcal{H}[f]$ is the map

$$\mathcal{H}[f] : \mathbb{C} \times \mathbb{SO}_2 \to \mathbb{C}$$

where $\mathbb{SO}_2$ is the space of all rotation matrices k_θ of positive determinant determined by an angle θ, sending each pair (s, k_θ) to the integral

$$\int_{\mathbb{H}_2} f(z)\mathrm{Im}(k_\theta(z))^{\bar{s}}\, dz$$

The *inverse Helgason-Fourier transform* is the map

$$\mathcal{H}^{-1}[\phi] : \mathbb{H}_2 \to \mathbb{C}$$

sending each z to the integral

$$\int_{\mathbb{R}}\int_{\mathbb{SO}_2} \phi(1/2 + it, k_\theta)\mathrm{Im}(k_\theta(z))^{1/2+it} {}^{t\ \tanh(t)\ d\theta\ dt}/_{8\pi^2}$$

For each f, we have the identities

$$\mathcal{H}^{-1}[\mathcal{H}[f]] = f, \quad \|f\|_2 = \|\mathcal{H}[f]\|_2. \tag{11}$$

In a certain sense, the operation $\mathcal{H}$ takes convolutions to products in the following sense. Let g denote a compactly supported density on $\mathbb{SL}_2$ that is $\mathbb{SO}_2$-*invariant* in the sense that $g(axb) = g(x)$ for all $a, b \in \mathbb{SO}_2$. Then each g passes to a well-defined density on $\mathbb{H}_2 = \mathbb{SL}_2/\mathbb{SO}_2$, which, by abuse of notation, is also written as g. As a function on $\mathbb{SL}_2$, the convolution $g * f$ can be defined as the density on $\mathbb{H}_2$ defined by

$$(g * f)(z) = \int_{\mathbb{SL}_2} g(m)f(m^{-1}z)dm,$$

where the integral is taken with respect to the Haar measure on $\mathbb{SL}_2$, the measure on $\mathbb{SL}_2$ that is unique up to scaling and invariant under multiplication on the left or right by an element. Then for all f and g,

$$\mathcal{H}[g * f] = \mathcal{H}[g]\mathcal{H}[f].$$

Moreover, the Helgason-Fourier transform and its inverse each send real-valued functions to real-valued functions.

B EFFICIENT COMPUTATION OF THE TEST STATISTIC

We compute our test statistics as follows. Given a pair of graphs G_1 and G_2 (of possibly varying size), we use generalized multidimensional scaling to obtain coordinates for G_1 and G_2, functions

$$\phi_1 : V_1 \to \mathbb{H}_2, \quad \phi_2 : V_2 \to \mathbb{H}_2$$

from the vertices V_1 of G_1 and V_2 of G_2. In our power tests, we generate our graphs G_1 and G_2 by sampling 100 points from two different densities on $\mathbb{H}_2$ and connecting those points according to the Heaviside step function, as outlined in Figure 3. Very rarely, generalized multidimensional scaling fails in the sense that $\cosh$ applied to the distance matrix does not have two negative eigenvalues. When such failure occurs during our power tests, we simply generate two new graphs of 100 nodes from appropriate densities on $\mathbb{H}_2$ and attempt the embedding algorithm once again.

We then let f_i be the generalized kernel density estimator (5) on $\mathbb{H}_2$ determined by the points $\phi_i(V_i)$, where $n = \#\phi_i(V_i)$, $T = n^{-1/6}$, and $h = {}^1/_{n+100}$ (experience shows that for the n values with which we are working, this choice of bandwidth h works best.) Thus the network models we estimate for G_1 and G_2 are the continuous latent space models $\hat{P}_1$ and $\hat{P}_2$ respectively determined by f_1 and f_2. The test statistic $d^* = d(\hat{P}_1, \hat{P}_2)$ we compute is

$$\|f_1 - f_2\|_2 = \int_{-T}^{+T}\int_0^{2\pi} (\mathcal{H}[f_2] - \mathcal{H}[f_1])^2 {}^{t\tanh t}/_{8\pi^2}\, d\theta\, dt.$$

approximated by averaging the integrand over 100 uniformly chosen pairs $(t, \theta) \in [-T, +T] \times [0, 2\pi)$. The integrand itself is computed as follows. The Helgason-Fourier transform $\mathcal{H}[f_i]$ of (5) is the function

$$\mathcal{H}[f_i] : \mathbb{R} \times \mathbb{SO}_2 \to \mathbb{R}$$

defined by the rule

$$\mathcal{H}[f_i](t, \theta) = \frac{1}{n}\sum_{i=1}^{n} \mathrm{Im}(k_\theta(Z_i))^{1/4 - t^2} e^{1/4 - t^2} \tag{12}$$