A Probabilistic Logic for Reasoning about Uncertain Temporal Information

Dragan Doder
Computer Science and Communication
University of Luxembourg
6, rue Coudenhove-Kalergi L-1359 Luxembourg
dragan.doder@uni.lu

Zoran Ognjanović
Mathematical Institute
Serbian Academy of Sciences and Arts
Kneza Mihaila 36, 11000 Belgrade, Serbia
zorano@mi.sanu.ac.rs

Abstract

The main goal of this work is to present the proof-theoretical and model-theoretical approach to a probabilistic logic which allows reasoning about temporal information. We extend both the language of linear time logic and the language of probabilistic logic, allowing statements like "A will always hold"and "the probability that A will hold in next moment is at least the probability that B will always hold," where A and B are arbitrary statements. We axiomatize this logic, provide corresponding semantics and prove that the axiomatization is sound and strongly complete. We show that the problem of deciding decidability is PSPACE-complete, no worse than that of linear time logic.

1 INTRODUCTION

The study of temporal logics started with the seminal work of Arthur Prior [Prior, 1957]. Temporal logics are designed in order to analyze and reason about the way that systems change over time, and have been shown to be a useful tool in describing behavior of an agent's knowledge base, for specification and verification of programs, hardware, protocols in distributed systems etc. [Emerson, 1990, Emerson, 1995]. In many practical situations the temporal information is not known with certainty. A typical example is formal representation of information about tracking moving objects with GPS systems, in the case in which the locations or the identities of the objects are not certainly known [Grant et al., 2010].

Many different tools are developed for representing, and reasoning with, uncertain knowledge. One particular line of research concerns the formalization in terms of probabilistic logic. After Nilsson [Nilsson, 1986] gave a procedure for probabilistic entailment which, given probabilities of premises, calculates bounds on the probabilities of the derived sentences, researchers from the field started investigation about formal systems for probabilistic reasoning. [Fagin et al., 1990] provided a finitary axiomatization for reasoning about linear combinations of probabilities, and they proved weak completeness (every consistent formula is satisfiable). Their formulas are Boolean combinations of the expressions of the form $r_1 w(\alpha_1) + \ldots + r_n w(\alpha_n) \geq r_{n+1}$, where w is the probability operator and α_i's are propositional formulas. The semantics of the logic use finitely additive probabilities, since σ-additivity cannot be expressed by a formula of their language.

In this paper, we extend the approach from [Fagin et al., 1990]. We start with the propositional linear time logic (LTL) [Gabbay et al., 1980] with the "next" operator $\bigcirc$ and "until" operator U. The meaning of the formula $\bigcirc\alpha$ is "α holds in the next time instance", and $\alpha U \beta$ we read "α holds in every time instance until β holds". We apply the probabilistic operator w to the formulas of LTL and define probabilistic formulas using the linear combinations, like in [Fagin et al., 1990]. In our logic there are two types of formulas, LTL formulas and probabilistic formulas, with the requirement that if an LTL formula is true, then its probability is equal to 1.

The main technical challenge in axiomatizing such a logic lies in the fact that the set of models of the formula $\alpha U \beta$ can be represented as a countable union of models of temporal formulas which are pairwise disjoint. As a consequence, finitely additive semantics is obviously not appropriate for such a logic, and we propose σ-additive semantics for the logic. On the other hand, expressing σ-additivity with an axiom would require infinite disjunctions, and the resulting logic would be undecidable. We shown in Section 3.1 that any finitary axiomatic system wouldn't be complete for the σ-additive semantics.

In order to overcome this problem, we axiomatize our language using infinitary rules of inference. Thus, in this work the term "infinitary" concerns the meta language only, i.e., the object language is countable and the formulas are finite, while only proofs are allowed to be infinite. We prove that our axiomatization is sound and strongly complete (every consistent set of formulas is satisfiable). We also prove that the logic is decidable, and we show that the satisfiability problem is $PSPACE$-complete, no harder then satisfiability for LTL.

There are several logics which combine time and probability in different ways [Guelev, 2000, Haddawy, 1996, Halpern and Pucella, 2006, Hansson and Jonsson, 1994, Ognjanovic, 2006, Shakarian et al., 2011]. However, to the best of our knowledge, this is the first complete axiomatization for the σ-additive probabilistic semantics.

2 THE LOGIC PL_{LTL}: SYNTAX AND SEMANTICS

We present the syntax and semantics of the logic for probabilistic reasoning about linear time formulas, that we denote by PL_{LTL}. The logic contains two types of formulas: formulas of LTL without probabilities, and the linear weight formulas in the style of [Fagin et al., 1990], with weights applied to temporal formulas.

In order to give semantics to the formulas, we first briefly review some probability theory [Ash and Doléans-Dade, 1999]. If $W \neq \emptyset$, then H is an algebra of subsets of W, if it is a set of subsets of W such that:

(a) $W \in H$,
(b) if $A, B \in H$, then $W \setminus A \in H$ and $A \cup B \in H$.

A function $\mu : H \longrightarrow [0,1]$ is a (σ-additive) probability measure, if the following conditions hold:

(1) $\mu(W) = 1$,
(2) $\mu(\bigcup_{i\in\omega} A_i) = \sum_{i\in\omega} \mu(A_i)$, whenever $A, A_i \in H$ and $A_i \cap A_j = \emptyset$ for all $i \neq j$.

For W, H and μ described above, the triple $\langle W, H, \mu\rangle$ is called a probability space. A function $\mu : H \longrightarrow [0,1]$ is a finitely additive probability measure, if the condition

(3) $\mu(A \cup B) = \mu(A) + \mu(B)$, whenever $A \cap B = \emptyset$.

holds, instead of (2). We also say that an algebra H is a σ-algebra, if $\bigcup_{i\in\omega} A_i \in H$ whenever $A_i \in H$ for every $i \in \omega$.

For a finitely additive μ, the condition (2) is equivalent to the condition

(2') $\mu(\bigcup_{i\in\omega} A_i) = \lim_{n\to+\infty} \mu(\bigcup_{i=0}^{n} A_i)$.

We will actually use (2') in the axiomatization of our logic (see the inference rule R6).

2.1 SYNTAX

First we introduce LTL formulas. Suppose that $\mathcal{P}$ is a nonempty finite set of propositional letters. We denote the elements of $\mathcal{P}$ with p and q, possibly with subscripts.

Definition 1 (LTL formula) *An* LTL formula *is any formula built from propositional letters from $\mathcal{P}$, using the Boolean connectives $\neg$ and $\wedge$, and the temporal operators $\bigcirc$ and U.*

We use For_{LTL} for the set of all state formulas and denote arbitrary LTL formulas by α, β and γ, possibly with subscripts.

We use $\neg$ and $\wedge$ as the primitive connectives, while other Boolean connectives ($\rightarrow, \vee, \leftrightarrow$) can be introduced as usual. We also define other LTL operators F (sometime) and G (always) as abbreviations: $F\alpha$ is $\top U\alpha$, and $G\alpha$ is $\neg F\neg\alpha$. Note that we use the strong version of U, which means that if $\alpha U\beta$ holds in a path, then β must hold eventually.

Example 1 *The expression*

$$\bigcirc(p \wedge q) \rightarrow (pU\neg q)$$

is an example of LTL formula. Its meaning is "if both p and q hold in the next moment, then p will hold until q becomes false".

Semantics for LTL formulas consists of the set of *paths*, where a path is a ω-structure in $\mathcal{P}$, of the form $\sigma = s_0, s_1, s_2, \ldots$ Here s_i, called *the i-th time instance of* σ, is a subset of $\mathcal{P}$, and $p \in s_i$ represent the propositional letter p being true at time i in σ. We denote the set of all paths with $\overline{\Sigma}$. In the rest of the paper, we use the following abbreviations:

- $\sigma_{\geq i}$ is the path $s_i, s_{i+1}, s_{i+2}, \ldots$
- σ_i is the state s_i.

The evaluation function[1] $v : \overline{\Sigma} \times For_{LTL} \longrightarrow \{0,1\}$ is defined recursively as follows:

- if $p \in \mathcal{P}$, then $v(\sigma, p) = 1$ iff $p \in \sigma_0$,
- $v(\sigma, \neg\alpha) = 1$ iff $v(\sigma, \alpha) = 0$,
- $v(\sigma, \alpha \wedge \beta) = 1$ iff $v(\sigma, \alpha) = 1$ and $v(\sigma, \beta) = 1$,
- $v(\sigma, \bigcirc\alpha) = 1$ iff $v(\sigma_{\geq 1}, \alpha) = 1$,
- $v(\sigma, \alpha U\beta) = 1$ iff there is some $i \in \omega$ such that $v(\sigma_{\geq i}\beta) = 1$, and for each $j \in \omega$, if $0 \leq j < i$ then $v(\sigma_{\geq j}, \beta) = 1$.

[1] In the literature, the evaluation of LTL formulas in paths is usually given in terms of satisfiability relation $\models$. We do not follow this notation, because in this paper we use $\models$ to denote satisfiability of formulas in PL_{LTL}-structures.

We say that α is true in the path σ, if $v(\sigma, \alpha) = 1$.

Now we introduce the probabilistic formulas. By $\mathcal{Q}$ we denote the set of rational numbers. First we define the probabilistic terms.

Definition 2 (Probabilistic term) *A probabilistic term is any expression of the form*

$$r_1 w(\alpha_1) + \ldots + r_k w(\alpha_k) + r_{k+1},$$

where k is a positive integer, and for all $i \leq k+1$, $\alpha_i \in For_{LTL}$ and $r_i \in \mathcal{Q}$.[2]

We use f and g, possibly subscripted, to denote probabilistic terms.

Definition 3 (Probabilistic formula) *A basic probabilistic formula is any formula of the form $f \geq r$, where f is a probabilistic term and $r \in \mathcal{Q}$. The set For_P of probabilistic formulas is the smallest set containing all basic probabilistic formulas, closed under Boolean connectives.*

We denote by ϕ, ψ and θ (possibly with indices) the elements of For_P. To simplify notation, we define the following abbreviations: $f \geq g$ is $f - g \geq 0$, $f \leq g$ is $g \geq f$, $f < g$ is $\neg f \geq g$, , $f > g$ is $\neg f \leq g$ and $f = g$ is $f \geq g \wedge f \leq g$.

Example 2 *The expression*

$$w(p \vee q) = w(\bigcirc p) \rightarrow w(Gq) \leq \frac{1}{2}$$

is a probabilistic formula. Its meaning is "if the probability that either p or q hold in this moment is equal to the probability that p will hold in the next moment, then the probability that q will always hold is at most one half".

Definition 4 (Formula) *The set For of all formulas of the logic PL_{LTL} is $For = For_{LTL} \cup For_P$.*

We denote arbitrary formulas by Φ and Ψ (possibly with subscripts). We denote by $\perp$ both $\phi \wedge \neg\phi$ and $\alpha \wedge \neg\alpha$, letting the context determines the meaning. Similarly, we use $\top$ for both LTL and probabilistic formulas.

Example 3 *The expression*

$$(p \vee \bigcirc q) \rightarrow w(p \vee \bigcirc q) = 1$$

is not a formula, since mixing LTL formulas and probabilistic formulas is not allowed, by Definition 4.

[2] In [Fagin et al., 1990], r_{k+1} does not appear in the definition of terms. We introduce it for the simpler presentation, when we introduce other formulas as abbreviations.

2.2 SEMANTICS

The semantics of the logic PL_{LTL} is based on the possible-world approach.

Definition 5 (PL_{LTL} structure) *A PL_{LTL} structure is a tuple $M = \langle W, H, \mu, \pi \rangle$ where:*

- *W is a nonempty set of worlds,*
- *$\langle W, H, \mu \rangle$ is a probability space, and*
- *$\pi : W \longrightarrow \overline{\Sigma}$ provides for each world $w \in W$ a path $\pi(w)$.*

For a PL_{LTL} structure $M = \langle W, H, \mu, \pi \rangle$, we define

$$[\alpha]_M = \{w \in W \mid v(\pi(w), \alpha) = 1\}.$$

We say that M is *measurable*, if $[\alpha]_M \in H$ for every $\alpha \in For_{LTL}$. We denote the class of all measurable PL_{LTL} structures with PL_{LTL}^{Meas}.

Now we define the satisfiability of a formula from For in a structure from PL_{LTL}^{Meas}.

Definition 6 (Satisfiability) *Let $M = \langle W, H, \mu, \pi \rangle$ be a PL_{LTL} structure. We define the satisfiability relation $\models \subseteq PL_{LTL}^{Meas} \times For$ recursively as follows:*

- *$M \models \alpha$ iff $v(\pi(w), \alpha) = 1$ for every $w \in W$,*
- *$M \models r_1 w(\alpha_1) + \ldots + r_k w(\alpha_k) \geq r$ iff $r_1 \mu([\alpha_1]_M) + \ldots + r_k \mu([\alpha_k]_M) \geq r$,*
- *$M \models \neg\phi$ iff $M \not\models \phi$,*
- *$M \models \phi \wedge \psi$ iff $M \models \phi$ and $M \models \psi$.*

Definition 7 (Model) *We say that $M \in PL_{LTL}^{Meas}$ is a model of Φ, if $M \models \Phi$. A formula Φ is valid, if $M \models \Phi$ holds for every $M \in PL_{LTL}^{Meas}$. We say that M is a model of a set of formulas T, and we write $M \models T$, iff $M \models \Phi$ for every $\Phi \in T$. A set of formulas T is satisfiable if there is M such that $M \models T$.*

Definition 8 (Entailment) *We say that the set of formulas T entails a formula Φ, and we write $T \models \Phi$, if all $M \in PL_{LTL}^{Meas}$, $M \models T$ implies $M \models \Phi$.*

For every $\alpha, \beta \in For_{LTL}$, let us denote by $\alpha \overline{U}_n \beta$ the formula

$$(\bigwedge_{k=0}^{n-1} \bigcirc^k \alpha) \wedge \bigcirc^n \beta,$$

and by $\alpha U_n \beta$ the formula $\bigvee_{k=0}^{n} \alpha \overline{U}_n \beta$.

Those formulas will play the important role in our axiomatization. Obviously, $v(\sigma, \alpha U \beta) = 1$ iff there is some $n \in \omega$ such that $v(\sigma, \alpha \overline{U}_n \beta) = 1$, and

$$[\alpha U \beta]_M = \bigcup_{n \in \omega} [\alpha \overline{U}_n \beta]_M. \tag{1}$$

Similarly,

$$[\alpha U \beta]_M = \bigcup_{n \in \omega} [\alpha U_n \beta]_M. \tag{2}$$

Since (1) follows directly from the definition of the evaluation function v, we will use it to properly axiomatize LTL part of our logic. On the other hand, (2) is more convenient for capturing σ-additivity.

3 The axiomatization of PL_{LTL}

In this section we provide an axiomatization for PL_{LTL}, which we denote by $AX_{PL_{LTL}}$. Let us first discuss some axiomatization issues. By (2) and σ-additivity, we obtain $\mu([\alpha U \beta]_M) = \mu(\bigcup_{n \in \omega} [\alpha U_n \beta]_M)$. Then we can see that the set

$$T = \{w(\alpha U \beta) > r\} \cup \{w(\alpha U_n \beta) \leq r | n \in \omega\}$$

is an unsatisfiable set of formulas. On the other hand, it is easy to check that every finite subset of T is satisfiable. In other words, the logic is not compact. It is known that, in this case, any finitary axiomatization would be incomplete [van der Hoek, 1997]. Here we use an infinitary rule (R6) to obtain completeness, and, in particular, to make the set T inconsistent. It turns that it is necessary (see the proof of Theorem 4) to introduce another infinitary rule (R4) to properly axiomatize LTL part of the logic, since the set of LTL formulas $\{\alpha U \beta\} \cap \{\neg(\alpha \overline{U}_n \beta) \mid n \in \omega\}$ is also an example of non-compactness.

3.1 THE AXIOMATIC SYSTEM $AX_{PL_{LTL}}$

the axiomatization $AX_{PL_{LTL}}$ contains 8 axioms and 6 rules of inference. We divide the axioms into 3 groups as given below.

Tautologies

A1. All instances of classical propositional tautologies for both LTL and probabilistic formulas.

Temporal axioms

A2. $\bigcirc(\alpha \to \beta) \to (\bigcirc\alpha \to \bigcirc\beta)$.

A3. $\neg \bigcirc \alpha \leftrightarrow \bigcirc\neg\alpha$.

A4. $\alpha U \beta \leftrightarrow \beta \vee (\alpha \wedge \bigcirc(\alpha U \beta))$.

Axioms for reasoning about linear inequalities

A5. All instances of valid formulas about linear inequalities.

Probabilistic axioms

A6. $w(\alpha) \geq 0$.

A7. $w(\alpha \wedge \beta) + w(\alpha \wedge \neg\beta) = w(\alpha)$.

A8. $w(\alpha \to \beta) = 1 \to w(\alpha) \leq w(\beta)$.

Inference rules

R1. From Φ and $\Phi \to \Psi$ infer Ψ (where either $\Phi, \Psi \in For_{LTL}$ or $\Phi, \Psi \in For_P$).

R2. From α infer $\bigcirc\alpha$.

R3. From α infer $w(\alpha) = 1$.

R4. From the set of premises

$$\{\gamma \to \neg(\alpha \overline{U}_n \beta) \mid n \in \omega\}$$

infer $\gamma \to \neg(\alpha U \beta)$.

R5. From the set of premises

$$\{\phi \to f \geq r - \frac{1}{n} \mid n \in \omega \setminus \{0\}\}$$

infer $\phi \to f \geq r$.

R6. From the set of premises

$$\{\phi \to w(\alpha U_n \beta) \leq r \mid n \in \omega\}$$

infer $\phi \to w(\alpha U \beta) \leq r$.

Let us briefly discuss the axiomatic system.
A1 and R1 allow propositional reasoning with all formulas from For.
The axioms A2–A4 are some standard axioms in various axiomatization of LTL. Although all the axiomatizations contain some additional axioms, we show in Lemma 1(1) that all the valid temporal formulas can be deduced in $AX_{PL_{LTL}}$. Moreover, by Lemma 2, A1–A4 together with R1,R2 and R4 make a strongly complete system for LTL. Note that we use the temporal necessitation R2 with the next operator, while the standard generalization can be derived, as it is shown in the proof of Lemma 1(1). The rule R4 is an infinitary rule that characterizes the until operator. It is similar to a rule from [Marinkovic et al., 2014], and it is necessary for the proof of σ-additivity.
The axiom A5 includes all valid formulas about linear inequalities. For example, $f + 1 \leq f + 2$ and $f + g =$

$g + f$ are instances of A5. A particular sound and complete axiomatization for Boolean combination is given in [Fagin et al., 1990], but, as it is pointed out there, any other axiomatization can be used.
The probabilistic axioms A6 and A7 correspond to non-negativity and finite additivity, respectively. They are two of the four axioms presented in [Fagin et al., 1990]. Other two axioms are theorems of $AX_{PL_{LTL}}$ (see Lemma 1). The rule R3 states that if we know that α holds, then we believe that it is true with probability 1. The rules R4–R6 are infinitary rules of inference. R4 and R6 are crucial for the proof of σ-additivity, while R5, ensures that the values of probability measures belong to the set of reals. R5 is a variant of a rule introduced in [Perovic et al., 2008].

Definition 9 (Proof) *A formula Φ is a* theorem *of the logic PL_{LTL}, ($\vdash \Phi$), if there is an at most countable sequence of formulas $\Phi_0, \Phi_1, \ldots, \Phi$, such that every Φ_i is an axiom, or it is derived from the preceding formulas by an inference rule.*
A formula Φ is deducible *from a set of formulas T ($T \vdash \Phi$) if there is an at most countable sequence of formulas $\Phi_0, \Phi_1, \ldots, \Phi$, such that every Φ_i is a theorem or a formula from T, or it is derived from the preceding formulas by one of the inference rules, excluding R2. The corresponding sequence $\Phi_0, \Phi_1, \ldots, \Phi$ is the* proof *of Φ from T.*

By the previous definition, application of the rule R2 is restricted to theorems only. Otherwise, any change during the time would be impossible. Note that the length of a proof (the number of formulas in the corresponding sequence) is any countable successor ordinal.

Definition 10 (Consistency) *A set of formulas T is consistent if there is no $\phi \in For_P$ such that $T \vdash \phi \wedge \neg\phi$, otherwise it is inconsistent. T is maximal consistent if it is consistent and for all $\Phi \notin T$, $T \cup \{\Phi\}$ is inconsistent.*

Next we make several observations about the notions of consistency and maximal consistency:

- If T is consistent, then there is no $\alpha \in For_{LTL}$ such that $T \vdash \alpha \wedge \neg\alpha$, since otherwise $T \vdash w(\alpha) = 1 \wedge w(\neg\alpha) = 1$ by R3, and $T \vdash w(\alpha) = 1 \wedge \neg w(\alpha) = 1$ by probabilistic axioms.

- Maximal consistency of T doesn't imply that for every $\alpha \in For_{LTL}$ either $T \vdash \alpha$ or $T \vdash \neg\alpha$. Indeed, suppose that $w(\alpha) = \frac{1}{2} \in T$ for some α. If $T \vdash \alpha$ or $T \vdash \neg\alpha$, then by R3 (and some probabilistic reasoning) we have $T \vdash w(\alpha) = 1$ or $T \vdash w(\alpha) = 0$, which would make T inconsistent. On the other hand, for a $\phi \in For_P$ we have either $T \vdash \phi$ or $T \vdash \neg\phi$ (see Lemma 1(4)).

- If T is consistent, then T is *deductively closed*, i.e., if $T \vdash \Phi$ then $\Phi \in T$.

3.2 SOME THEOREMS ABOUT $AX_{PL_{LTL}}$

It is straightforward to check that all the axioms of $AX_{PL_{LTL}}$ are valid, and that the rules of inference maintain the validity of formulas. Thus, we omit the proof of the following result.

Theorem 1 (Soundness) *The axiomatization $AX_{PL_{LTL}}$ is sound with respect to the class of models PL_{LTL}^{Meas}.*

Theorem 2 (Deduction theorem) *Let T be a set of formulas and let Φ and Ψ be two formulas such that either $\Phi, \Psi \in For_{LTL}$ or $\Phi, \Psi \in For_{LTL}$. Then $T \cup \{\Phi\} \vdash \Psi$ iff $T \vdash \Phi \to \Psi$.*

Proof. (sketch) We will prove the direction from right to left because the other direction is immediate from R1. We will use induction on the length of the inference. We will only consider the case when R6 is applied. Suppose that $T \cup \{\phi\} \vdash \psi \to w(\alpha U \beta) \leq r$ is obtained by R6. Then $T \cup \{\phi\} \vdash \psi \to w(\alpha U_n \beta) \leq r$ holds, by assumption, for every $n \in \omega$. Using induction hypothesis and reasoning as above, we have:
$T \vdash \phi \to (\psi \to w(\alpha U_n \beta) \leq r)$, for for every $n \in \omega$;
$T \vdash (\phi \wedge \psi) \to w(\alpha U_n \beta) \leq r$, for every $n \in \omega$;
$T \vdash (\phi \wedge \psi) \to w(\alpha U \beta) \leq r$, by R6;
$T \vdash \phi \to (\psi \to w(\alpha U \beta) \leq r)$.

Lemma 1

1. *If $v(\sigma, \alpha) = 1$ for all $\sigma \in \overline{\Sigma}$, then $\vdash \alpha$.*
2. $\vdash w(\top) = 1$
3. *If $T \vdash \alpha \leftrightarrow \beta$, then $T \vdash w(\alpha) = w(\beta)$*
4. *If T is maximal consistent then either $\phi \in T$ or $\neg\phi \in T$, for every $\phi \in For_P$.*

Proof. (1) If is sufficient to prove that all the axioms of any complete axiomatization of LTL (for example C1–C8 form [Reynolds, 2001]) are theorems of our logic, and that the standard Generalization rule "if α is a theorem, from α infer $G\alpha$" is derived rule in $Ax_{PL_{LTL}}$. As an ilustration, let us derive Generalization. If $\vdash \alpha$, applying rule R2 we obtain $\vdash \bigcirc^n \alpha$ for every $n \in \omega$. Using A3, we conclude $\vdash \neg \bigcirc^n \neg\alpha$ for every $n \in \omega$. Note that $\neg \bigcirc^n \neg\alpha$ can be writen as $\neg(\top \overline{U}_n \neg\alpha)$. Finally, applying R4 we obtain $\vdash \neg(\top U \neg\alpha)$, or, equivalently, $\vdash G\alpha$.
(2) Follows directly form R3.
(3) Apply R3, then A8.
(4) If $\phi \notin T$, then $T \cup \{\phi\} \vdash \bot$, by the maximality of T. By Theorem 2, we have $T \vdash \phi \to \bot$, so $T \vdash \neg\phi$. Similarly, if$\phi \notin T$, then $T \vdash \neg\phi$, which contradicts the assumption that T is consistent.

Let us comment the lemma. By (1), we can use all the standard theorems of LTL in our reasoning in PL_{LTL}. (2) is an

axiom for probabilistic reasoning from [Fagin et al., 1990]. (3) plays the crucial role in the construction of the canonical model in the next section. If we choose α and β to be propositional formulas and $T = \emptyset$, we obtain another axiom from [Fagin et al., 1990]. Thus, by (1)–(3), $AX_{PL_{LTL}}$ extends both temporal and probabilistic logic.
We use (4) in the proof of Theorem 5. We already pointed out that the same property doesn't hold for the LTL formulas. Note that we cannot copy the proof of (4) in LTL case, since we distinguish between the probabilistic contradiction and LTL contradiction (although we use $\bot$ in both cases).

4 THE COMPLETENESS OF PL_{LTL}

In this section we prove strong version of completeness theorem: "every consistent set of formulas has a model". We use a Henkin-like construction. First we extend a consistent set T of formulas to a maximal consistent set T^*, then we use T^* to define the corresponding structure M_{T^*}, and finally we prove that M_{T^*} is a model of T. For given T^*, we say that M_{T^*} is its canonical model.

4.1 LINDENBAUM'S LEMMA

Theorem 3 (Lindenbaum's lemma) *Every consistent set of formulas can be extended to a maximal consistent set.*

Proof.(sketch) Let T be a consistent set and let $\Phi_0, \Phi_1, \ldots$ be an enumeration of all formulas from For. We define the sequence of sets T_i, $i = 0, 1, 2, \ldots$ and the set T^* recursively as follows:

1. $T_0 = T$,
2. for every $i \geq 0$,
   (a) if $T_i \cup \{\Phi_i\}$ is consistent, then $T_{i+1} = T_i \cup \{\Phi_i\}$, otherwise
   (b) if Φ_i is of the form $\gamma \to \neg(\alpha U \beta)$, then $T_{i+1} = T_i \cup \{\gamma \to (\alpha \overline{U}_n \beta)\}$, where n is the smallest nonnegative integer such that T_{i+1} is consistent, otherwise
   (c) if Φ_i is of the form $\phi \to f \geq r$, then $T_{i+1} = T_i \cup \{\phi \to f < r - \frac{1}{n}\}$, where n is the smallest positive integer such that T_{i+1} is consistent, otherwise
   (d) if Φ_i is of the form $\phi \to w(\alpha U \beta) \leq r$, then $T_{i+1} = T_i \cup \{\phi \to w(\alpha U_n \beta) > r\}$, where n is the smallest nonnegative integer such that T_{i+1} is consistent, otherwise
   (e) $T_{i+1} = T_i$.
3. $T^\star = \bigcup_{i=0}^{\infty} T_i$.

First, using Theorem 2 one can prove that the set T^* is correctly defined, i.e., there exist n from the parts 2(b)–2(d) of the construction. Each T_i, $i > 0$ is consistent. The steps (1) and (2) of the construction ensure that $T^\star$ is maximal. Also, $T^\star$ obviously doesn't contain all formulas. Finally, one can show that $T^\star$ is deductively closed set, and as a consequence we obtain that $T^\star$ is consistent (otherwise it would contain $\bot$).

4.2 CANONICAL MODEL

Definition 11 (Canonical model) *For a maximal consistent set T^*, we define a PL_{LTL} structure as a tuple $M_{T^*} = \langle W, H, \mu, \pi \rangle$, such that:*

1. $W = \{\sigma \in \overline{\Sigma} \mid v(\sigma, \alpha) = 1$ *for all* $\alpha \in T^* \cap For_{LTL}\}$,
2. $H = \{[\alpha] \mid \alpha \in For_{LTL}\}$, *where* $[\alpha] = \{w \in W \mid v(w, \alpha) = 1\}$,
3. $\mu([\alpha]) = \sup\{r \in \mathcal{Q} \mid T^* \vdash w(\alpha) \geq r\}$, *for every* $\alpha \in For_{LTL}$,
4. $\pi(w) = w$ *for every* $w \in W$.

Now we show that M_{T^*} is a measurable PL_{LTL} structure. In the proof, we will use the following result.

Lemma 2 *The axioms A1–A4 and the inference rules R1, R2 and R4 form a strongly complete axiomatization for LTL.*

Proof. We need to show that every consistent set T of LTL formulas has a model, i.e., that there is σ such that $v(\sigma, \alpha) = 1$ for every $\alpha \in T$. Reasoning similarly as above, we can prove that Deduction theorem holds and that T can be extended to a maximal consistent set T^*. Now we work with LTL formulas only, and we can prove that for each α either $\alpha \in T^*$ or $\neg\alpha \in T^*$. Also, using the axiomatization it is straightforward to show that if T^* is maximal consistent set, then the set $T^*_n = \{\alpha \mid \bigcirc \alpha \in T^*\}$ is also maximal consistent.
For given T^* , we define the path $\sigma = s_0, s_1, \ldots$ by $s_i = \{p \in \mathcal{P} \mid T^*_i \vdash p\}$.
It is sufficient to prove that $v(\sigma, \gamma) = 1$ iff $T^* \vdash \gamma$, for every LTL formula γ. We use induction on the complexity of the formula. The only interesting case is when γ is of the form $\alpha U \beta$.
$v(\sigma, \gamma) = 0$ iff $v(\sigma, \neg(\alpha U \beta)) = 1$
iff for all $n \in \omega$, it is not the case that $v(\sigma_{\geq n}, \beta) = 1$ and for all $k < n$, $v(\sigma_{\geq k}, \alpha) = 1$
iff for all $n \in \omega$, it is not the case that $T^*_n \vdash \beta$ and for all $k < n$, $T^*_k \vdash \alpha$ (by induction hypothesis)
iff for all $n \in \omega$, it is not the case that $T^* \vdash \bigcirc^n \beta$ and for all $k < n$, $T^* \vdash \bigcirc^k \alpha$

iff for all $n \in \omega$, $T^* \vdash \neg(\alpha \overline{U}_n \beta)$ (by the maximal consistency of T^*)
iff $T^* \vdash \neg(\alpha U \beta)$ (by R4).

Theorem 4 *For every maximal consistent set T^*, $M_{T^*} \in PL_{LTL}^{Meas}$.*

Proof. First we need to show that the definition is correct. The set $\{[\alpha] \mid \alpha \in For_{LTL}\}$ is an algebra of subsets of W, since $W = [\top]$, $W \setminus [\alpha] = [\neg\alpha]$ and $[\alpha] \cup [\beta] = [\alpha \vee \beta]$. We also need to check that μ is correctly defined, i.e., that if $[\alpha] = [\beta]$ then $\mu([\alpha]) = \mu([\beta])$. From $[\alpha] = [\beta]$ we conclude that if σ is a path such that $v(\sigma, \gamma) = 1$ for all $\gamma \in T^* \cap For_{LTL}$, then $v(\sigma, \alpha \leftrightarrow \beta) = 1$. From Lemma 2 we obtain $T^* \vdash \alpha \leftrightarrow \beta$. Consequently, $T^* \vdash w(\alpha) = w(\beta)$ by Lemma 1(3), so $\mu([\alpha]) = \mu([\beta])$. Obviously $\mu(W) = \mu([\top]) = 1$ by Lemma 1(2). Similarly, using A6 we conclude that μ is nonnegative, and using A7 we conclude that μ is a finitely additive probability measure on A. We need to prove that μ is σ-additive.
Let $H_{\overline{\Sigma}} = \{[\alpha]_{\overline{\Sigma}} \mid \alpha \in For_{LTL}\}$, where $[\alpha]_{\overline{\Sigma}} = \{\sigma \in \overline{\Sigma} \mid v(w, \alpha) = 1\}$. By $For_{LTL}^{\bigcirc}$ we denote the set of all LTL formulas in which $\bigcirc$ is the only temporal operator (i.e. there are no appearances of U). We also introduce the set $A = \{[\alpha] \mid \alpha \in For_{LTL}^{\bigcirc}\}$. Using the same argument as above, we can show that the sets $H_{\overline{\Sigma}}$ and A are two algebras of subsets of $\overline{\Sigma}$. Similarly as in the definition of M_{T^*}, we define μ^* on $H_{\overline{\Sigma}}$ by

$$\mu^*([\alpha]_{\overline{\Sigma}}) = \sup\{r \in \mathcal{Q} \mid T^* \vdash w(\alpha) \geq r\}.$$

Reasoning as above, we conclude that μ^* is a finitely additive measure. We also use the same symbol μ^* to denote the restriction of μ^* to A. We actually want to show that μ^* is σ-additive on A. It is sufficient to show that if $B = \bigcup_{n \in \omega} B_i$, where $B, B_i \in A$, then there is n such that $B = \bigcup_{n=0}^{\omega} B_i$.
If $2^{\mathcal{P}}$ denotes the set of subsets of $\mathcal{P}$, note that $\overline{\Sigma} = 2^{\mathcal{P}} \times 2^{\mathcal{P}} \times 2^{\mathcal{P}} \times \ldots$ If we assume discrete topology on the finite set $2^{\mathcal{P}}$ and the induced product topology on $\overline{\Sigma}$, then $\overline{\Sigma}$ is a compact space as a product of compact spaces.[3]
By definition of evaluation function v, we obtain that for every $\alpha \in For_{LTL}^{\bigcirc}$ there exist $n \in \omega$ (for example n is the number of appearances of $\bigcirc$) and $S \subseteq (2^{\mathcal{P}})^n$ such that $[\alpha]_{\overline{\Sigma}} = S \times 2^{\mathcal{P}} \times 2^{\mathcal{P}} \times \ldots$ are Note that the sets of the form $S \times 2^{\mathcal{P}} \times 2^{\mathcal{P}} \times \ldots$, where $S \subseteq (2^{\mathcal{P}})^n$ for some $n \in \omega$, are clopen (both closed and open) sets in product topology. Thus, each $[\alpha]_{\overline{\Sigma}} \in A$ is a clopen set in $\overline{\Sigma}$. Now assume that $[\alpha]_{\overline{\Sigma}} = \bigcup_{n \in \omega} [\alpha_n]_{\overline{\Sigma}}$, where $\alpha \in For_{LTL}^{\bigcirc}$ and $\alpha_n \in For_{LTL}^{\bigcirc}$ for every $n \in \omega$. The set $\{[\alpha_n]_{\overline{\Sigma}} \mid n \in \omega\}$ is an open cover of the closed subset $[\alpha]_{\overline{\Sigma}}$ of the compact space $\overline{\Sigma}$, so there is a finite subcover $\{[\alpha_{n_1}]_{\overline{\Sigma}}, \ldots, [\alpha_{n_1}]_{\overline{\Sigma}}\}$ of $[\alpha]_{\overline{\Sigma}}$. Thus, μ^* is σ-additive on A.

[3]For the basic notions and results about the topology used here we refer the reader to [Kechris, 1995]

Let F be the σ-algebra generated by A. Since $[\alpha U \beta]_{\overline{\Sigma}} = \bigcup_{n \in \omega} [\alpha U_n \beta]_{\overline{\Sigma}}$, we can show that $[\alpha]_{\overline{\Sigma}} \in F$ for every $\alpha \in For_{LTL}$, using the induction on the number of appearances of U in α. Thus, $H_{\overline{\Sigma}} \subseteq F$. By Caratheodory's extension theorem (see [Ash and Doléans-Dade, 1999]), there is a unique σ-additive probability measure ν on F which coincide with μ^* on A. We will actually show that μ^* is the restriction of ν to $H_{\overline{\Sigma}}$, i.e., that $\mu^*([\alpha]_{\overline{\Sigma}}) = \nu([\alpha]_{\overline{\Sigma}})$ for all $\alpha \in For_{LTL}$, using the induction on the number of appearances of U in α. Indeed, $\nu([\alpha]_{\overline{\Sigma}}) = \nu(\bigcup_{n \in \omega} [\alpha U_n \beta]_{\overline{\Sigma}}) = \lim_{k \to +\infty} \nu(\bigcup_{n=1}^{k} [\alpha U_n \beta]_{\overline{\Sigma}}) = \lim_{k \to +\infty} \mu^*(\bigcup_{n=1}^{k} [\alpha U_n \beta]_{\overline{\Sigma}}) = \mu^*([\alpha U \beta]_{\overline{\Sigma}})$. Here we used σ-additivity of ν, the induction hypothesis and, in the last step, the definition of μ^* and R6.
Thus, μ^* is a σ-additive probability measure on $H_{\overline{\Sigma}}$. Note that we have that $\mu^*([\alpha]_{\overline{\Sigma}}) = 1$ whenever $T^* \vdash \alpha$, by R3. Thus, $\mu^*(W) = \mu^*(\bigcap_{\alpha: T^* \vdash \alpha} [\alpha]_{\overline{\Sigma}}) = 1$, by σ-additivity of μ^*.
Note that $[\alpha] = [\alpha]_{\overline{\Sigma}} \cap W$, so $H \subseteq F$. Let $\overline{\mu}$ be the σ-additive probability measure on H induced by μ^* by

$$\overline{\mu}([\alpha]) = \overline{\mu}([\alpha]_{\overline{\Sigma}} \cap W) = \mu^*([\alpha]_{\overline{\Sigma}}).$$

Note that $\mu^*(W) = 1$ implies $\mu^*([\alpha]_{\overline{\Sigma}}) = \mu^*([\alpha]_{\overline{\Sigma}} \cap W)$, so $\mu^*([\alpha]) = \nu([\alpha])$. By definitions of μ and μ^* it follows that μ and ν coincide. Thus, μ is σ-additive.
We showed that M_{T^*} is a PL_{LTL} structure. Finally, note that $[\alpha] = [\alpha]_{M_{T^*}}$, by the choice of π, so $M_{T^*} \in PL_{LTL}^{Meas}$.

Now we can prove the main result of this section.

4.3 COMPLETENESS THEOREM

Theorem 5 (Strong completeness) *A set of formulas $T \subseteq For$ is consistent iff it is satisfiable.*

Proof. The direction from right to left follows from the soundness of the axiomatization $AX_{PL_{LTL}}$. For the other direction, we need to show that a consistent set of formulas T has a model. First we extend T to a maximal consistent set T^*, and we construct the canonical model M_{T^*}. We will show that M_{T^*} is a model of T^*, and, consequently, a model of T. It is sufficient to prove that for all $\Phi \in For$, $T^* \vdash \Phi$ iff $M_{T^*} \models \Phi$.
If $\Phi = \alpha \in For_{LTL}$. If $\alpha \in T^*$, then by the definition of W from M_{T^*}, $M_{T^*} \models \alpha$. Conversely, if $M_{T^*} \models \alpha$, by Lemma 2, $\alpha \in T^*$.
If $\Phi \in For_P$, we proceed by induction on the complexity of Φ.
Let $\Phi = f \geq r$. If $f = r_1 w(\alpha_1) + \ldots + r_k w(\alpha_k) + r_{k+1}$, we can show, using the properties of supremum, that

$$r_1 \mu([\alpha_1]) + \ldots + r_k \mu([\alpha_k]) + r_{k+1} = \sup\{s \mid T^* \vdash f \geq s\}.$$

If we suppose that $f \geq r \in T^*$, then $r \leq \sup\{s \mid T^* \vdash f \geq s\}$, so $M_{T^*} \models f \geq r$. For the other direction, assume that $M_{T^*} \models f \geq r$. Then $M_{T^*} \not\models f < r$. If $f < r \in T^*$,

then, reasoning as above, we conclude $M_{T^*} \models f < r$, a contradiction. By Maximality of T^*, we obtain $f \geq r \in T^*$.
If $\Phi = \neg\phi$, then $M_{T^*} \models \neg\phi$ iff $M_{T^*} \not\models \phi$ iff $\phi \notin T^*$ iff $\neg\phi \in T^*$, by maximality of T^*.
If $\Phi = \phi \wedge \psi$, then $M_{T^*} \models \phi \wedge \psi$ iff $M_{T^*} \models \phi$ and $M_{T^*} \models \phi$ iff $\phi, \psi \in T^*$ iff $\phi \wedge \psi \in T^*$, by maximality of T^*.

As it is well known, the alternative formulation of Completeness theorem, stated below, follows directly from the previous result.

Theorem 6 *If $T \subseteq For$ and $\Phi \in For$, then $T \models \Phi$ iff $T \vdash \Phi$.*

5 THE DECIDABILITY OF PL_{LTL}

[Sistla and Clarke, 1985] proved that the logic LTL is decidable, and they showed that the problem of deciding whether an LTL formula is satisfiable in a path is $PSPACE$-complete. Note that if α is not satisfiable in any path, then by Definition 6 it is not satisfiable in the logic PL_{LTL}. On the other hand, if there is a path σ such that $v(\sigma, \alpha) = 1$, then we can define a measurable structure $M = \langle W, H, \mu, \pi\rangle$, such that $W = \{w\}$ is a singleton and $\pi(w) = \sigma$ (note that in that case the range of μ is $\{0, 1\}$). Obviously, $v(\pi(w), \alpha) = 1$ for every $w \in W$, so $M \models \alpha$. Thus, we proved that the satisfiability problem of LTL formulas for the logic PL_{LTL} is $PSPACE$-complete.

Now let us consider the satisfiability of a formula $\varphi \in For_P$. Let $For_B(\varphi)$ denote the set of all basic probabilistic formulas which appear in φ. Suppose that the formula $\varphi \in For_P$ is given in the complete disjunctive normal form (CDNF),. i.e., $\varphi = \bigvee_{i=1}^{m} \varphi_i$, where each φ_i is a conjunction of the formulas from $For_B(\varphi)$ or their negations, using all elements of $For_B(\varphi)$, i.e. the number of conjuncts of each φ_i is $|For_B(\varphi)|$. Note that the disjunction φ is satisfiable iff at least one of its disjuncts φ_i is satisfiable.

Thus, we focus on satisfiability of the formulas of the form

$$\bigwedge_{k=1}^{|For_B(\varphi)|} \psi_k, \tag{3}$$

where each ψ_k is a basic formula or its negation. In the following, we assume that a formula of the form (3) is given, and we denote by F the set of its conjuncts $\{\psi_k \mid k = 1, \ldots, |For_B(\varphi)|\}$.

For a LTL formula α, by $Subfor(\alpha)$ we denote the set of its subformulas. If $For_{LTL}(F)$ is the set of all LTL formulas which appear in at least one element of F (under the scope of probability operator w), let $Subfor = \bigcup_{\alpha \in For_{LTL}(F)} Subfor(\alpha)$. Let us consider the formulas of the form

$$\bigwedge_{k=1}^{|Subfor|} \beta_k, \tag{4}$$

where each β_k belongs to $Subfor \cup \{\neg\beta \mid \beta \in Subfor\}$, and each subformula of α appears exactly once (negated or not). Obviously the conjunction of any two different formulas of the form (4) is a contradiction, while the disjunction of all such formulas is a tautology. This enables us to translate the satisfiability problem to the problem of finding a solution of a system of inequalities. First, note that there are $2^{|Subfor|}$ formulas of the form (4). First we eliminate the formulas which are not satisfiable in LTL, using the procedure from [Sistla and Clarke, 1985]. Suppose that there are ℓ formulas which are satisfiable ($\ell \leq 2^{|Subfor|}$). We denote those formulas by $\alpha_1, \ldots, \alpha_\ell$.

For any formula $\alpha \in For_{LTL}(F)$ we have that $\alpha \in Subfor$. Consequently, α appears in each conjunction α_k, negated or not. Since $\bigvee_{k=1}^{\ell} \alpha_k$ is a tautology, there is a unique set of indices $I_\alpha \subseteq \{1, \ldots, \ell\}$ such that $\alpha \leftrightarrow \bigvee_{i \in I_\alpha} \alpha_i$ is a tautology. Let Γ_α be the corresponding set $\{\alpha_i \mid i \in I_\alpha\}$. Using the probabilistic axioms and Lemma 1(3), we obtain

$$\vdash w(\alpha) = \sum_{\alpha_i \in \Gamma_\alpha} w(\alpha_i). \tag{5}$$

Now, we can transform every formula $\psi \in F$ of the form $r_1 w(\gamma_1) + \ldots + r_k w(\gamma_k) \geq r_{k+1}$ to the equivalent formula

$$r_1 \sum_{\alpha_i \in \Gamma_{\gamma_1}} w(\alpha_i) + \ldots + r_k \sum_{\alpha_i \in \Gamma_{\gamma_k}} w(\alpha_i) \geq r_{k+1}. \tag{6}$$

Thus, we obtain that a measurable structure $M = \langle W, H, \mu, \pi\rangle$ satisfies ψ if and only if

$$r_1 \sum_{\alpha_i \in \Gamma_{\gamma_1}} \mu([\alpha_i]) + \ldots + r_k \sum_{\alpha_i \in \Gamma_{\gamma_k}} \mu([\alpha_i]) \geq r_{k+1}. \tag{7}$$

Similarly, if ψ from F is a negation of a basic probabilistic formula, then it is of the form $r_1 w(\gamma_1) + \ldots + r_k w(\gamma_k) < r_{k+1}$, which give us the similar condition for satisfiability of ψ under M:

$$r_1 \sum_{\alpha_i \in \Gamma_{\gamma_1}} \mu([\alpha_i]) + \ldots + r_k \sum_{\alpha_i \in \Gamma_{\gamma_k}} \mu([\alpha_i]) < r_{k+1}. \tag{8}$$

Let denote by x_i the probability of the formula α_i in a potential model $M = \langle W, H, \mu, \pi\rangle$ of the formula (3), i.e., $x_i = \mu([\alpha_i])$ each $i \in \{1, \ldots, \ell\}$.

Let F_{pos} be the set of basic probabilistic formulas from F, and let F_{neg} be the set of formulas from F which are negations of basic probabilistic formulas. For given $\psi \in F_{pos}$ of the form $r_1 w(\gamma_1) + \ldots + r_k w(\gamma_k) \geq r_{k+1}$ we define the inequality $Ineq(\psi)$, obtained by (7), as

$$Ineq(\psi): \; r_1 (\sum_{i:\alpha_i \in \Gamma_{\gamma_1}} x_i) + \ldots + r_k (\sum_{i:\alpha_i \in \Gamma_{\gamma_k}} x_i) \geq r_{k+1}.$$

In the same way we define $Ineq(\psi)$ for $\psi \in F_{neg}$ of the form $r_1 w(\gamma_1) + \ldots + r_k w(\gamma_k) < r_{k+1}$ as

$$Ineq(\psi): \ r_1(\sum_{i:\alpha_i \in \Gamma_{\gamma_1}} x_i) + \ldots + r_k(\sum_{i:\alpha_i \in \Gamma_{\gamma_k}} x_i) < r_{k+1}.$$

Then the formula (3) is satisfiable iff the following sentence of the language of real closed fields is satisfiable:

$$\begin{aligned} \exists x_1 \ldots \exists x_\ell \quad & \big(\bigwedge_{k=1}^{\ell} (x_k \geq 0) \\ & \wedge \sum_{k=1}^{\ell} x_k = 1 \\ & \wedge \bigwedge_{\psi \in F} Ineq(\psi) \big). \end{aligned}$$

The sentence represents a nonlinear system of linear inequalities: the first line represents non-negativity of probability measures; the second line represents the condition $\mu(W) = \mu([\top]) = \sum_{k=1}^{\ell} \mu([\alpha_k]) = 1$. The third line represent the conditions (7) and (8). Obviously, if the system doesn't have a solution, there is no μ which satisfies (3). If the system has the solution $(x_1, \ldots, x_\ell) = (c_1, \ldots, c_\ell)$, then we can construct $M = \langle W, H, \mu, \pi \rangle$ which satisfies (3) in the following way: $W = \{w_1, \ldots w_\ell\}$, $\pi(w_i)$ is any path σ such that $v(\sigma, \alpha_i) = 1$, H is the set of all subsets of W and μ is determined by the condition $\mu(\{w_i\}) = c_i$.

Since the theory of real closed fields is decidable, our logic is decidable as well. Moreover, note that the above sentence is an existential sentence. Thus, we can use Canny's decision procedure from [Canny, 1988]. Since the procedure decides satisfiability of the formula in PSPACE, we conclude that satisfiability of probabilistic formulas is in PSPACE as well.

Thus, in both probabilistic and LTL case there is a procedure which decides satisfiability of the formula in $PSPACE$. Since $PSPACE$ is also a lower bound in the case of LTL formulas, we proved the following result.

Theorem 7 *The problem of deciding whether a formula of the logic PL_{LTL} is satisfiable in a measurable structure from PL_{LTL}^{Meas} is $PSPACE$-complete.*

6 CONCLUSION

In this paper, we introduced the logic PL_{LTL} for probabilistic reasoning about temporal information. The language contains both LTL formulas and probabilistic formulas in the style of [Fagin et al., 1990], with the difference that the probabilistic operator w is now applied to LTL formulas. We propose an axiomatization for the logic and prove strong completeness. Since the semantical relationship between the operators "next" and "until" explicitly requires σ-additive semantics, the axiomatization contains infinitary rules of inference. We show that the satisfiability problem is $PSPACE$-complete, no harder then satisfiability for LTL.

It seems that combining any standard finitary axiomatizatin of LTL with the axiomatization from [Fagin et al., 1990] could be extended to a weakly (but not strongly) complete axiomatization for a finitely additive restriction of our logic, which would be convenient for possible applications. On the other hand, we believe that our infinitary rules of inference can be represented using schemes (similarly as quantifiers in the first order logic are abbreviations for the infinite conjunctions and disjunctions), so that some of infinitary proofs might be finitary represented and used in automated reasoning.

Some probabilistic LTL's were motivated by the need to analyze probabilistic programs and stochastic systems [Donaldson and Gilbert, 2008, Feldman, 1984, Hansson and Jonsson, 1994, Kozen, 1985, Lehmann and Shelah, 1982]. In some of them, probabilistic operators are not explicitly mentioned in the formulas, while in the others it is possible to directly express probabilities. Our logic allows one to quantify runs satisfying some properties. In this paper we restrict our attention to theoretical issues (e.g., worst case complexity), while the possible applications (e.g., heuristic procedures for satisfiability checking) are left for the future work.

Acknowledgements

This work was supported by the National Research Fund (FNR) of Luxembourg through project PRIMAT, and by the Serbian Ministry of Education and Science through projects ON174026 and III44006.

We wish to thank the anonymous UAI referees whose comments and suggestions helped us to improve the paper. We also wish to thank Marc van Zee for his help.

References

[Ash and Doléans-Dade, 1999] Ash, R. B. and Doléans-Dade, C. A. (1999). *Probability & Measure Theory, Second Edition*. Academic Press, 2 edition.

[Canny, 1988] Canny, J. F. (1988). Some algebraic and geometric computations in PSPACE. In *Proceedings of the 20th Annual ACM Symposium on Theory of Computing, May 2-4, 1988, Chicago, Illinois, USA*, pages 460–467.

[Donaldson and Gilbert, 2008] Donaldson, R. and Gilbert, D. (2008). A monte carlo model checker for probabilistic ltl with numerical constraints. Technical report, University of Glasgow, Department of Computing Science.

[Emerson, 1990] Emerson, A. E. (1990). Temporal and modal logic. pages 995–1072.

[Emerson, 1995] Emerson, E. A. (1995). Automated temporal reasoning about reactive systems. In *Logics for Concurrency - Structure versus Automata (8th Banff Higher Order Workshop, August 27 - September 3, 1995, Proceedings)*, pages 41–101.

[Fagin et al., 1990] Fagin, R., Halpern, J. Y., and Megiddo, N. (1990). A logic for reasoning about probabilities. *Inf. Comput.*, 87(1/2):78–128.

[Feldman, 1984] Feldman, Y. A. (1984). A decidable propositional dynamic logic with explicit probabilities. *Information and Control*, 63(1/2):11–38.

[Gabbay et al., 1980] Gabbay, D. M., Pnueli, A., Shelah, S., and Stavi, J. (1980). On the temporal basis of fairness. In *Conference Record of the Seventh Annual ACM Symposium on Principles of Programming Languages, Las Vegas, Nevada, USA, January 1980*, pages 163–173.

[Grant et al., 2010] Grant, J., Parisi, F., Parker, A., and Subrahmanian, V. S. (2010). An agm-style belief revision mechanism for probabilistic spatio-temporal logics. *Artif. Intell.*, 174(1):72–104.

[Guelev, 2000] Guelev, D. P. (2000). Probabilistic neighbourhood logic. In *Formal Techniques in Real-Time and Fault-Tolerant Systems, 6th International Symposium, FTRTFT 2000, Pune, India, September 20-22, 2000, Proceedings*, pages 264–275.

[Haddawy, 1996] Haddawy, P. (1996). A logic of time, chance, and action for representing plans. *Artif. Intell.*, 80(1-2):243–308.

[Halpern and Pucella, 2006] Halpern, J. Y. and Pucella, R. (2006). A logic for reasoning about evidence. *J. Artif. Intell. Res. (JAIR)*, 26:1–34.

[Hansson and Jonsson, 1994] Hansson, H. and Jonsson, B. (1994). A logic for reasoning about time and reliability. *Formal Asp. Comput.*, 6(5):512–535.

[Kechris, 1995] Kechris, A. S. (1995). *Classical Descriptive Set Theory (Graduate Texts in Mathematics) (v. 156)*. Springer, 1 edition.

[Kozen, 1985] Kozen, D. (1985). A probabilistic PDL. *J. Comput. Syst. Sci.*, 30(2):162–178.

[Lehmann and Shelah, 1982] Lehmann, D. J. and Shelah, S. (1982). Reasoning with time and chance. *Information and Control*, 53(3):165–198.

[Marinkovic et al., 2014] Marinkovic, B., Ognjanovic, Z., Doder, D., and Perovic, A. (2014). A propositional linear time logic with time flow isomorphic to ω^2. *J. Applied Logic*, 12(2):208–229.

[Nilsson, 1986] Nilsson, N. J. (1986). Probabilistic logic. *Artif. Intell.*, 28(1):71–87.

[Ognjanovic, 2006] Ognjanovic, Z. (2006). Discrete linear-time probabilistic logics: Completeness, decidability and complexity. *J. Log. Comput.*, 16(2):257–285.

[Perovic et al., 2008] Perovic, A., Ognjanovic, Z., Raskovic, M., and Markovic, Z. (2008). A probabilistic logic with polynomial weight formulas. In *Foundations of Information and Knowledge Systems, 5th International Symposium, FoIKS 2008, Pisa, Italy, February 11-15, 2008, Proceedings*, pages 239–252.

[Prior, 1957] Prior, A. (1957). *Time and Modality*. Clarendon Press, Oxford.

[Reynolds, 2001] Reynolds, M. (2001). An axiomatization of full computation tree logic. *J. Symb. Log.*, 66(3):1011–1057.

[Shakarian et al., 2011] Shakarian, P., Parker, A., Simari, G. I., and Subrahmanian, V. S. (2011). Annotated probabilistic temporal logic. *ACM Trans. Comput. Log.*, 12(2):14.

[Sistla and Clarke, 1985] Sistla, A. P. and Clarke, E. M. (1985). The complexity of propositional linear temporal logics. *J. ACM*, 32(3):733–749.

[van der Hoek, 1997] van der Hoek, W. (1997). Some considerations on the logic pfd˜. *Journal of Applied Non-Classical Logics*, 7(3).